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Weak typed Böhm theorem on IMLL

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This paper is dedicated to the memory of Reijiro Kurata

Abstract

In the Böhm theorem workshop on Crete, Zoran Petric called Statman's "Typical Ambiguity theorem" the *typed Böhm theorem*. Moreover, he gave a new proof of the theorem based on set-theoretical models of the simply typed lambda calculus.

In this paper, we study the linear version of the typed Böhm theorem on a fragment of Intuitionistic Linear Logic. We show that in the multiplicative fragment of intuitionistic linear logic without the multiplicative unit **1** (for short IMLL) the weak typed Böhm theorem holds. The system IMLL exactly corresponds to the linear lambda calculus with multiplicative pairing. The system IMLL also exactly corresponds to the free symmetric monoidal closed category without the unit object. As far as we know, our separation result is the first one with regard to these systems in a purely syntactical manner.

Keywords: Linear logic; Separation; Proof nets; Graph isomorphisms

1. Introduction

In [6], Dosen and Petric called Statman's "Typical Ambiguity theorem" [16] the *typed Böhm theorem*. Moreover, they gave a new proof of the theorem based on set-theoretical models of the simply typed lambda calculus.

In this paper, we study the linear version of the typed Böhm theorem on intuitionistic multiplicative Linear Logic without the multiplicative unit **1** (for short IMLL). We consider the typed version of the following statement:

> There are two different closed $\beta\eta$-normal terms $\underline{0}$ and $\underline{1}$ such that if s and t are closed untyped normal λ-terms, and $s \neq_{\beta\eta} t$ then, there is a context $C[]$ such that
>
> $$C[s] =_{\beta\eta} \underline{0} \quad \text{and} \quad C[t] =_{\beta\eta} \underline{1}.$$

We call the statement the *weak untyped Böhm theorem*. In this paper, we show that the typed version of the weak Böhm theorem holds in IMLL.

The theorem is nontrivial because the system IMLL is rather weak in expressibility. Hence, a careful analysis on IMLL proof nets is needed. The system IMLL exactly corresponds to the linear lambda calculus with multiplicative

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pairing. A version of the linear lambda calculus can be found in [12]. The system IMLL also exactly corresponds to the free symmetric monoidal closed category without the unit object (see [12]). As far as we know, the result we prove in this paper is the first one with regard to these systems in a purely syntactical manner.

On the other hand, we call the following statement the *strong untyped Böhm theorem*:

> For any untyped λ-terms a and b, if s and t are closed untyped normal λ-terms, and $s \neq_{\beta\eta} t$, then there is a context $C[]$ such that
>
> $$C[s] =_{\beta\eta} a \quad \text{and} \quad C[t] =_{\beta\eta} b.$$

We could not prove the typed version of the statement in the system IMLL. But so far we proved the typed version of the statement w.r.t. a very limited fragment including additive connectives of Linear Logic (see Section 6). Also note that the weak statement and the strong statement are trivially equivalent in the untyped λK-calculus (i.e., the usual λ-calculus) and in the simply typed λ-calculus (if type instantiation is allowed) because both systems allow unrestricted weakening.

In addition, this paper includes the following several technical novelties:

(1) A characterization of an equality of IMLL proof nets based on graph isomorphisms in terms of the notion of *extended main paths*.
(2) We prove that the number of the IIMLL formulas with order less than 4 and with exactly two closed proof nets is essentially two. That is to say, they are $\mathbf{PBool} = p \multimap (p \multimap p) \multimap (p \multimap p) \multimap p$ and $\mathbf{PBool}' = p \multimap p \multimap (p \multimap p \multimap p) \multimap p$. Any of the other IIMLL formulas with the same property belongs to the same equivalence class as **PBool** or $\mathbf{PBool}'$, where the equivalence relation is about permutations of subformulas.
(3) We prove that all the representable functions by the closed normal proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ are exactly constant functions and (positive and negative) projections with n-arguments.
(4) We derive recursive equations which give the number of the closed normal proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$. We also derive recursive equations which give the number of the closed normal proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ that represents a (positive or negative) projection.

Related works. Our work is obviously based on that of [16] (see also [14,15,13]). As we said before, however, our result cannot be derived directly from that of [16], mainly because of lack of unrestricted weakening in IMLL. It is also interesting that unlike ours, the separability result of [16] cannot be obtained simply by substituting a type which has only two closed normal terms: a type which should be instantiated depends on the maximal number of occurrences of variables if you want to restrict the type to have only a finite number of closed terms, since the simply typed lambda calculus allows unrestricted contraction. Of course, you can choose a type which has infinitely many closed terms like the Church integer. But IMLL does not have such a type.

On the other hand, recently, some works [5,9,17,18,11] other than [6] have been also done on similar topics to the typed Böhm theorem. However, the system with which [9] and [5,6] dealt is the simply typed lambda calculus or the free cartesian closed category, not IMLL. The works of [17,18,11] are technically completely different from ours.

The structure of the paper. Section 2 introduces a definition of IMLL proof nets and an equality on them based on *graph isomorphisms*. Moreover it gives a characterization of the equality based on the notion of *extended main paths*. Section 3 proves that given a pair $\langle \Theta_1, \Theta_2 \rangle$ such that Θ_1 and Θ_2 are closed IMLL proof nets with the same conclusion and $\Theta_1 \neq \Theta_2$, there is a wrapping net $C[]$, which is an analogue to a context in λ-calculus, such that $C[\Theta_1]$ and $C[\Theta_2]$ are closed IIMLL proof nets with the same conclusion with an order less than 4 and $C[\Theta_1] \neq C[\Theta_2]$. This means that we can transform complex proof nets to simpler proof nets in an injective and internal manner. The key point of our separation result is type instantiation by a partial boolean type $\mathbf{PBool} = p \multimap (p \multimap p) \multimap (p \multimap p) \multimap p$.

Section 4 gives a characterization of the closed normal proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ ($n \geq 1$). The characterization says that the set is exactly the set of constant functions and (positive and negative) projections. Section 5 proves that given a pair $\langle \Theta_1, \Theta_2 \rangle$ such that Θ_1 and Θ_2 are closed IIMLL proof nets with the same conclusion

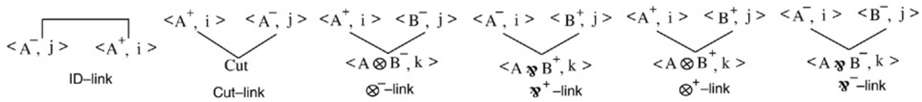


Fig. 1. IMLL links.

with an order less than 4 and $\Theta_1 \neq \Theta_2$, there is a wrapping net $C[]$ such that $C[\Theta_1] = \underline{0}$ and $C[\Theta_2] = \underline{1}$. This is established by finding an assignment in *the second-order linear term system*. Then we complete our proof of the weak typed Böhm theorem on IMLL. Section 6 discusses extensions of our result to IMLL with the multiplicative constant **1**, MLL, and IMLL with additives.

2. The IMLL systems

In this section, we present intuitionistic multiplicative proof nets. We also call these *IMLL proof nets*.

Definition 1 (*Negation-free MLL Formulas*). Negation-free MLL formulas (or simply formulas) (F) are inductively constructed from atomic formulas (P) and logical connectives:

- $P = p$
- $F = P \mid F \otimes F \mid F \invamp F$.

In this paper, we only consider formulas with one propositional variable p. All the results in this paper can be easily extended to the general case with denumerable propositional variables, since we just substitute p for these propositional variables.

Definition 2 (*IMLL Formulas*). An IMLL formula is a pair $\langle A, pl \rangle$ where A is a negation-free MLL formula and pl is an element of $\{+, -\}$, where $+$ and $-$ are called Danos–Regnier polarities. A formula $\langle A, pl \rangle$ is written as A^{pl}. A formula with $+$ (resp. $-$) polarity is called a $+$-formula or positive formula (resp. $-$-formula or negative formula).

Definition 3 (*IIMLL Formulas*). The set of IIMLL formulas, which is a subset of the set of IMLL formulas, is defined inductively:

(1) p^+ and p^- are IIMLL formulas;
(2) if A^+ and B^- are IIMLL formulas, then $A \otimes B^-$ is an IIMLL formula;
(3) if A^- and B^+ are IIMLL formulas, then $A \invamp B^+$ is an IIMLL formula.

Definition 4 (*Indexed IMLL Formulas*). An indexed IMLL formula is a pair $\langle F, i \rangle$, where F is an IMLL formula and i is a natural number.

Fig. 1 shows the links we use in this paper. We call each link in Fig. 1 *an IMLL link*. In Fig. 1,

(1) In ID-link, $\langle A^+, i \rangle$ and $\langle A^-, j \rangle$ are called conclusions of the link.
(2) In Cut-link, $\langle A^+, i \rangle$ and $\langle A^-, j \rangle$ are called premises of the link.
(3) In $\otimes^-$-link (resp. $\invamp^+$-link) $\langle A^+, i \rangle$ (resp. $\langle A^-, i \rangle$) is called the left premise, $\langle B^-, j \rangle$ (resp. $\langle B^+, j \rangle$) the right premise and $\langle A \otimes B^-, k \rangle$ (resp. $\langle A \invamp B^+, k \rangle$) the conclusion of the link.
(4) In $\otimes^+$-link (respectively $\invamp^-$-link), $\langle A^+, i \rangle$ (resp. $\langle A^-, i \rangle$) is called the left premise, $\langle B^+, j \rangle$ (resp. $\langle B^-, j \rangle$) the right premise and $\langle A \otimes B^+, k \rangle$ (resp. $\langle A \invamp B^-, k \rangle$) the conclusion of the link.

Remark. Links are constituents of proof structures defined immediately below (and also of proof nets). In a proof structure, several links with the same kind and the same premises and conclusions may occur. Hence a link in a proof structure means an occurrence of the link. An occurrence of a link can be considered as a link with an index, i.e., a pair $\langle L_0, i \rangle$, where L_0 is a link and i is a natural number. Similarly in a proof structure a premise or a conclusion of a link occurrence denotes a *formula occurrence* in the proof structure. Such a formula occurrence also can be considered as a formula with an index, i.e., an indexed IMLL formula (see Section 2 of [7]). But note that such an indexing is local at the proof structure: this means that essentially the same two proof structures (and also two proof nets) can have different indexings. Therefore in order to define an equality of proof nets we must care about the issue. The problem is still the same even if we adopt Danos–Regnier style proof nets [4]. But since Danos–Regnier style proof nets are familiar directed labeled graphs, it is easy to define homomorphisms between them. So in order to define an equality of normal proof nets, we use a variant of Danos–Regnier style proof nets.

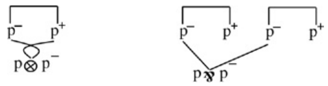


Fig. 2. Two examples of IMLL proof structures.

Definition 5 (*IMLL Proof Structures*). Let $\mathcal{F}$ be a set of IMLL formula occurrences, i.e., a set of indexed IMLL formulas, and $\mathcal{L}$ be a set of IMLL link occurrences. The pair $\Theta = \langle \mathcal{F}, \mathcal{L} \rangle$ is an IMLL proof structure if Θ satisfies the following conditions:

(1) for any $\langle F_0, i \rangle$ and $\langle F_0', j \rangle$ in $\mathcal{F}$, if $i = j$, then $F_0 = F_0'$ (i.e., in $\mathcal{F}$, each element has a different index);
(2) for each formula occurrence $F \in \mathcal{F}$, if F is a premise of a link occurrence $L \in \mathcal{L}$ then L is unique, i.e., F is not a premise of any other link $L' \in \mathcal{L}$;
(3) for each formula occurrence $F \in \mathcal{F}$, there is a unique link occurrence $L \in \mathcal{L}$ such that F is a conclusion of L.

Remark. In the following, when we discuss proof structures or proof nets, in many cases, we conveniently forget indexings for them, because such information is superfluous in many cases. Moreover, when we draw proof structures or proof nets, we also forget such indexings, because locative information in such drawings plays an indexing.

We say that in $\Theta = \langle \mathcal{F}, \mathcal{L} \rangle$, a formula occurrence $F \in \mathcal{F}$ is a conclusion of Θ if for any $L \in \mathcal{L}$, F is not a premise of L.

It is well-known that a proof structure does not necessarily correspond to a sequent calculus proof. For example, two IMLL proof structures in Fig. 2 do not have the corresponding sequent calculus proofs. The following sequentializability is a judgement on the correspondence.

Definition 6 (*Sequentializability*). An IMLL proof structure $\Theta = \langle \mathcal{F}, \mathcal{L} \rangle$ is sequentializable if any of the following conditions holds:

(1) $\mathcal{L} = \{L\}$ and L is an ID-link;
(2) there is a $⅋^{pl}$-link $L \in \mathcal{L}$ such that the conclusion $A ⅋ B^{pl}$ of L is a conclusion of Θ and $\langle \mathcal{F} - \{A ⅋ B^{pl}\}, \mathcal{L} - \{L\} \rangle$ is sequentializable;
(3) there is a $\otimes^{pl}$-link $L \in \mathcal{L}$ and there are two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ and two subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ such that (a) the conclusion $A \otimes B^{pl}$ of L is a conclusion of Θ, (b) $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \{A \otimes B^{pl}\}$, (c) $\mathcal{L} = \mathcal{L}_1 \uplus \mathcal{L}_2 \uplus \{L\}$, and (d) $\langle \mathcal{F}_1, \mathcal{L}_1 \rangle$ (respectively $\langle \mathcal{F}_2, \mathcal{L}_2 \rangle$) is an IMLL proof structure and sequentializable, where $\uplus$ denotes the disjoint union operator;
(4) there is a Cut-link $L \in \mathcal{L}$ and there are two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ and two subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ such that (a) $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, (b) $\mathcal{L} = \mathcal{L}_1 \uplus \mathcal{L}_2 \uplus \{L\}$, and (c) $\langle \mathcal{F}_1, \mathcal{L}_1 \rangle$ (respectively $\langle \mathcal{F}_2, \mathcal{L}_2 \rangle$) is an IMLL proof structure and sequentializable.

Definition 7 (*IMLL Proof Nets*). An IMLL proof structure Θ is an IMLL proof net if Θ is sequentializable.

From the definition, we can easily prove the following proposition.

Proposition 1. *An IMLL proof net Θ has exactly one positive conclusion.*

Proof. Since Θ is sequentializable, Θ has the corresponding IMLL sequent calculus proof with the same conclusions. The corresponding sequent calculus proof has exactly one positive conclusion. □

An IMLL proof net Θ is an *IIMLL proof net* if the conclusions of any ID-link in Θ are IIMLL formulas and Θ has neither any $\otimes^+$-link occurrence nor $⅋^-$-link occurrence. We can easily prove any formula occurrence in an IIMLL proof is an IIMLL formula by induction on the number of links.

Next we give a graph-theoretic characterization of IMLL proof nets, following [8], because we use this in the proofs of Proposition 10, Lemmas 4, 7 and 10. The characterization was first proved in [7] and then an improvement was given in [3]. In order to characterize IMLL proof nets among IMLL proof structures, we introduce *Danos–Regnier graphs*. Let Θ be an IMLL proof structure. We assume that we are given a function S from the set of all occurrences of $⅋$-links in Θ to $\{0, 1\}$. Such a function is called a *switching function* for Θ. Then the Danos–Regnier graph Θ_S for Θ and S is an undirected graph such that

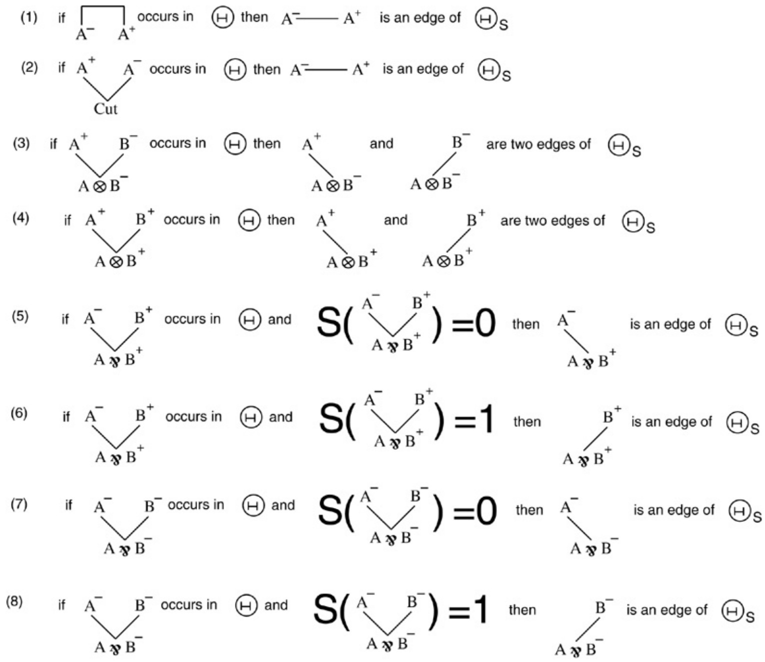


Fig. 3. The rules for the generation of the edges of a Danos–Regnier graph Θ_S.

(1) the nodes are all the formula occurrences in Θ, and
(2) the edges are generated by the rules of Fig. 3.

Theorem 1 *([7] and [3]). An IMLL proof structure Θ is an IMLL proof net iff for each switching function S for Θ, the Danos–Regnier graph Θ_S is acyclic and connected.*

Next we define reduction on IMLL proof nets. Fig. 4 shows the rewrite rules we use in this paper. The ID and multiplicative rewrite rules are usual ones. The multiplicative η-expansion is the usual η-expansion in Linear Logic. We denote the reduction relation defined by these five rewrite rules by $\rightarrow^*$. The one step reduction of $\rightarrow^*$ is denoted by $\rightarrow$. Note that we can easily prove that if Θ is an IMLL proof net and $\Theta \rightarrow \Theta'$, then Θ' is also an IMLL proof net (for example, see [8]). In the following subsection we show that strong normalizability and confluence w.r.t. $\rightarrow$ holds. Hence without mention, we identify an IMLL proof net with the normalized net.

Abbreviations. In the following we use an abbreviation using linear implication $\multimap$ instead of $\parr$ in order to relate our IMLL formulas to usual IMLL formulas in the linear lambda calculus (for example, in [12]).

(1) $\mathrm{abb}(A^+) = \mathrm{sabb}(A^+)^+ \quad \mathrm{abb}(A^-) = \mathrm{sabb}(A^-)^-$
(2) $\mathrm{sabb}(p^-) = \mathrm{sabb}(p^+) = p$
(3) $\mathrm{sabb}(A \otimes B^-) = \mathrm{sabb}(A^+) \multimap \mathrm{sabb}(B^-) \quad \mathrm{sabb}(A \parr B^+) = \mathrm{sabb}(A^-) \multimap \mathrm{sabb}(B^+)$
(4) $\mathrm{sabb}(A \otimes B^+) = \mathrm{sabb}(A^+) \otimes \mathrm{sabb}(B^+) \quad \mathrm{sabb}(A \parr B^-) = \mathrm{sabb}(A^-) \otimes \mathrm{sabb}(B^-)$.

For example, $\mathrm{abb}(p \parr (((p \otimes p) \parr (p \otimes p)) \parr p)^+)$ is $p \multimap (((p \multimap p) \otimes (p \multimap p)) \multimap p)^+$. An IMLL formula A^ϵ (where $\epsilon = +$ or $-$) is an *IIMLL formula* if $\mathrm{abb}(A^\epsilon)$ does not have any occurrences of $\otimes$-connectives. Note that only IIMLL formulas occur in an IIMLL proof net. We identify an IMLL formula A^ϵ with $\mathrm{abb}(A^\epsilon)$, where $\epsilon = +$ or $-$. The notation is a little bit confusing: for example, $\mathrm{abb}(p \parr p^-) = p \otimes p^-$. This is due to the mismatch between the proof-nets notation and the linear lambda calculus notation. However, from surrounding contexts, i.e., from whether $\parr$ or $\multimap$ is used, we can easily judge which notation is adopted. When we draw proof-nets, we mainly use abbreviated formulas.

2.1. Strong normalizability and confluence on the IMLL system

We believe that these two theorems are folklore. We just give the following proofs because of a request by an anonymous referee. The strong normalizability is almost trivial. The confluence on IMLL is more complicated because

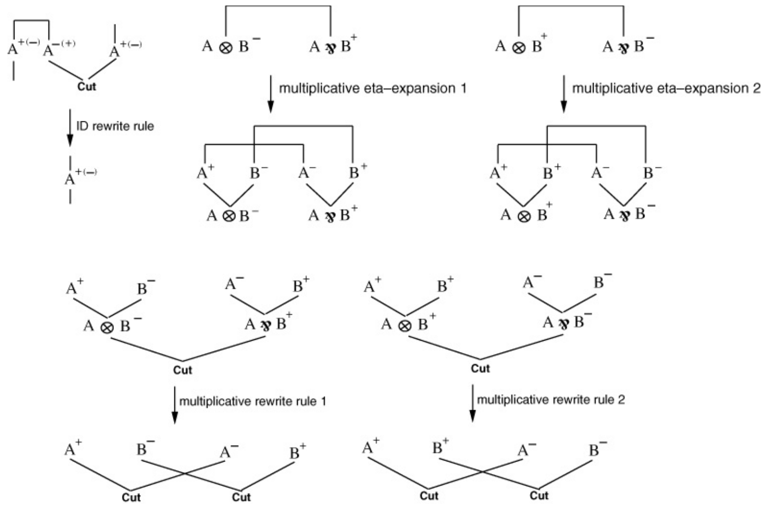


Fig. 4. The rewrite rules we use in this paper.

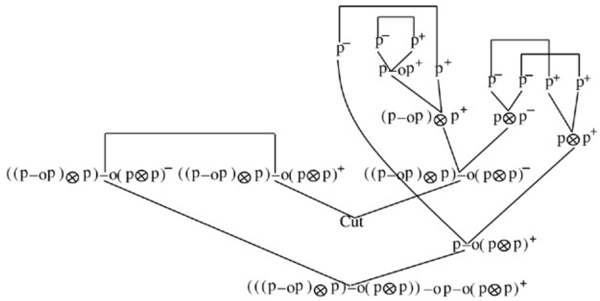


Fig. 5. An example of IMLL proof nets with Cut-links Θ_1.

in the IMLL with the multiplicative η-expansion one-step confluence does not hold unlike the IMLL without the rewrite rule. But we do not think that the proofs that we give here are difficult to understand. If you have no doubt about the strong normalizability and confluence on the IMLL system, you can skip this subsection.

Definition 8 (*The SN Size of an ID-link and the SN Size of a Cut-link*). The SN size of an ID-link is the size of a conclusion, that is, the number of the occurrences of logical connectives in the conclusion. Note that the choice between a conclusion and the other conclusion is indifferent. Also note that the SN size of an ID-link with two atomic formulas as the conclusions is 0. The SN size of a Cut-link is the size of a premise plus 1. With regard to the SN size of a Cut-link, the same remark about the choice between a premise and the other premise as that of an ID-link is also applied. Also note that the SN size of a Cut-link with two atomic formulas as the premises is 1.

Definition 9 (*The SN Size of an IMLL Proof net*). The SN size of an IMLL proof net Θ is the sum of the SN sizes of all the occurrences of Cut-links and ID-links in Θ.

Proposition 2 (*Strong Normalizability on the IMLL System*). *Let Θ be an IMLL proof net. Θ is strong normalizing.*

Proof. Let $\Theta \to \Theta'$. Then in any case where Θ reduces to Θ' by a rule in Fig. 4, we can easily see the SN size of Θ' is less than that of Θ. $\square$

For example, the SN size of Θ_1 in Fig. 5 is 9. Then $\Theta_1 \to \Theta_2$ by the ID rewrite rule, where Θ_2 is the IMLL proof net of Fig. 6. The SN size of Θ_2 is 0. On the other hand $\Theta_1 \to \Theta_3$ by the multiplicative η-expansion 1, where Θ_3 is the IMLL proof net of Fig. 7. The SN size of Θ_3 is 8.

Next, we consider the confluence on the IMLL system.

Figs. 5–7 show a counterexample of one-step confluence in the IMLL system with the multiplicative η-expansion, since Θ_3 of Fig. 7 cannot reach Θ_2 of Fig. 6 exactly by one-step. Nevertheless, applying the multiplicative η-expansion

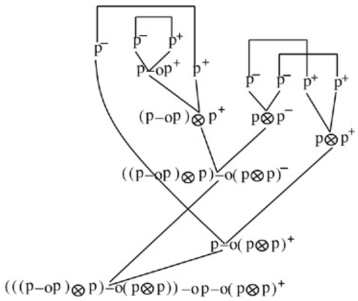


Fig. 6. The IMLL proof net Θ_2 obtained from Θ_1 by the ID rewrite rule.

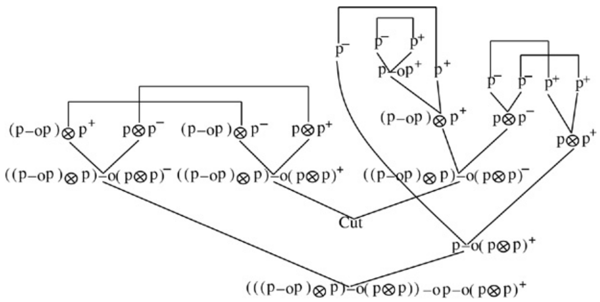


Fig. 7. The IMLL proof net Θ_3 obtained from Θ_1 by the multiplicative η-expansion 1.

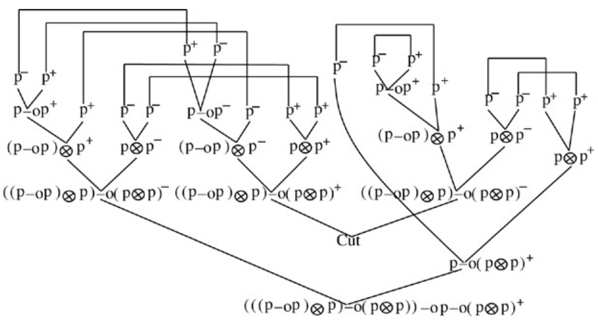


Fig. 8. The IMLL proof net Θ_4 obtained from Θ_3 by applying the multiplicative η-expansion three times.

three times to Θ_3, we can obtain Θ_4 and applying the multiplicative rewrite rule four times and the ID rewrite rule on atomic formulas five times to Θ_4 of Fig. 8, we can obtain Θ_2.

We also give another example. Figs. 9–11 also show a counterexample of one-step confluence in the IMLL system with the multiplicative η-expansion, since Θ_3' of Fig. 11 cannot reach Θ_2' of Fig. 10 exactly by one-step. Although we can obtain Θ_2' from Θ_3' by applying the multiplicative rewrite rule two times and the ID rewrite rule two times, we can also obtain Θ_2' from Θ_3', first obtaining Θ_4' of Fig. 12 from Θ_3' by the multiplicative η-expansion three times and second applying the multiplicative rule six times and the ID rule ten times.

In the following we formalize the intuition.

Definition 10 (*The Maximal η-Expansion of an ID-Link*). Let Θ be the IMLL proof net consisting of exactly one ID-link with A^+ and A^- as the conclusions. The maximal η-expansion of Θ is the IMLL proof net exactly with A^+ and A^- as the conclusions that does not have any ID-links except ID-links with only atomic conclusions obtained from Θ by applying multiplicative η-expansion rules maximally. We denote the η-expansion of Θ by η-expand(A^+, A^-).

Lemma 1. *Let Π be an IMLL proof net with a conclusion A^+ (respectively A^-). Then we let Θ be the IMLL proof net connecting Π and η-expand(A^+, A^-) by a Cut-link with the premises A^+ (respectively A^-) on Π and A^- (respectively A^+) on η-expand(A^+, A^-). Then there is an IMLL proof net Π' such that $\Pi \to^* \Pi'$ and*

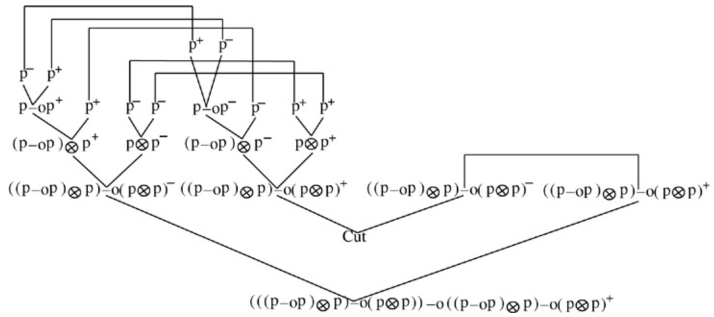


Fig. 9. Another example of IMLL proof nets with Cut-links Θ'_1.

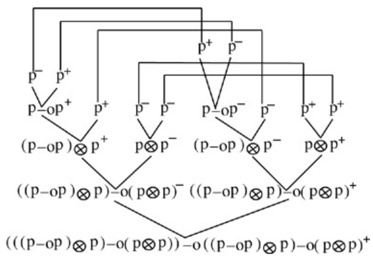


Fig. 10. The IMLL proof net Θ'_2 obtained from Θ'_1 by the ID rewrite rule.

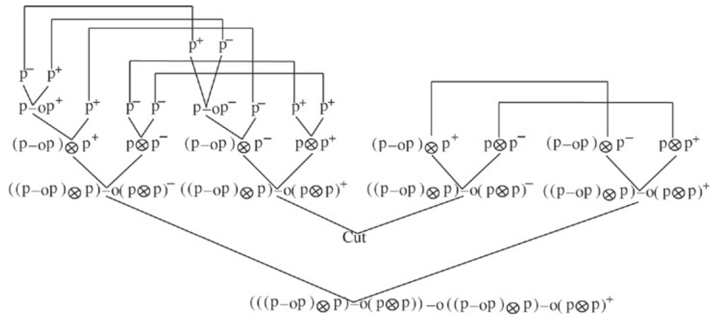


Fig. 11. The IMLL proof net Θ'_3 obtained from Θ'_1 by the multiplicative η-expansion 1.

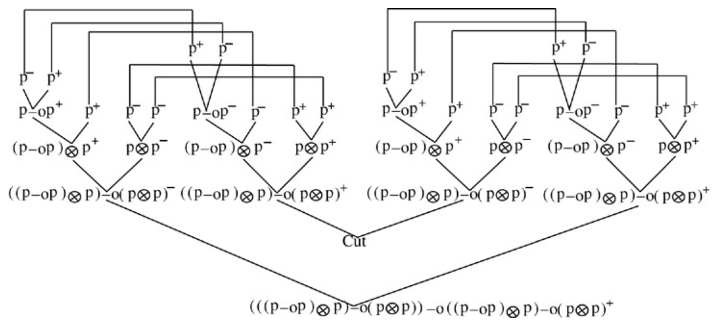


Fig. 12. The IMLL proof net Θ'_4 obtained from Θ'_3 by applying the multiplicative η-expansion three times.

$\Theta \to^* \Pi'$, where Π' is an IMLL proof net obtained from Π by applying the multiplicative η-expansion to some (possibly zero) subformula occurrences of A^+ (resp. A^-) of Π.

Proof. We prove this lemma by induction on A^+ (resp. A^-). We only consider A^+. The case of A^- is similar.

(1) The base step: the case where A^+ is an atomic formula p^+.

Then η-expand(A^+, A^-) is an IMLL proof net consisting exactly one ID-link with p^+, p^- as the conclusions. Then we can easily see that $\Theta \to \Pi$ by ID rewrite rule. So, it is OK to let Π' be Π.

(2) The induction step: the case where A^+ is not an atomic formula.

(a) the case where A^+ on Π is a conclusion of an ID-link:

Let Π' be the IMLL proof net obtained from Π by replacing the ID-link with η-expand(A^+, A^-). Then $\Pi \to^* \Pi'$. Moreover it is easily see to $\Theta \to \Pi'$ by the ID rewrite rule.

(b) the case where A^+ on Π is not a conclusion of an ID-link:

(i) the case where A^+ is a conclusion of $\parr$-link:

Then A^+ must have the form $A_1 \multimap A_2{}^+$. Let Θ' be the IMLL proof net such that $\Theta \to \Theta'$ by the multiplicative rewrite rule 1 w.r.t. $A^+ = A_1 \multimap A_2{}^+$ in Π and $A^- = A_1 \multimap A_2{}^-$ in η-expand(A^+, A^-). Let Θ'' be the IMLL proof net obtained from Θ' by removing the $\parr$-link with the conclusion $A_1 \multimap A_2{}^+$. Then Θ'' can be regarded as an IMLL proof net obtained from an IMLL proof net and η-expand(A_1^+, A_1^-) by connecting a Cut-link. Let Π_1 be the IMLL proof net obtained from Θ'' by removing η-expand(A_1^+, A_1^-) and its associated Cut-link. By inductive hypothesis, we can obtain an IMLL proof net Π_1' such that $\Pi_1 \to^* \Pi_1'$ and $\Theta'' \to^* \Pi_1'$, where Π_1' is obtained from Π_1 by applying the multiplicative η-expansion to some subformula occurrences of A_1^- of Π_1. Again Π_1' can be regarded as an IMLL proof net obtained from an IMLL proof net and η-expand(A_2^+, A_2^-) by connecting a Cut-link. Let Π_2 be the IMLL proof net obtained from Π_1' by removing η-expand(A_2^+, A_2^-) and its associated Cut-link. By inductive hypothesis again, we can obtain an IMLL proof net Π_2' such that $\Pi_2 \to^* \Pi_2'$ and $\Pi_1' \to^* \Pi_2'$, where Π_2' is obtained from Π_2 by applying the multiplicative η-expansion to some subformula occurrences of A_2^+ of Π_1. Finally let Π' be the IMLL proof net obtained from Π_2' by adding a $\parr$-link with the conclusion $A_1 \multimap A_2{}^+$. Then Π' is an IMLL proof net obtained from Π by applying the multiplicative η-expansion to some subformula occurrences of $A_1 \multimap A_2{}^+$ of Π, since $\Pi_0 \to^* \Pi_2 \to^* \Pi_2'$, where Π_0 is an IMLL proof net obtained from Π by removing the $\parr$-link with the conclusion $A^+ = A_1 \multimap A_2{}^+$. Hence $\Pi \to^* \Pi'$. Moreover it can be easily seen that $\Theta \to^* \Pi'$ since $\Theta'' \to^* \Pi_1' \to^* \Pi_2'$.

(ii) the case where A^+ is a conclusion of $\otimes$-link:

Then A^+ must have the form $A_1 \otimes A_2{}^+$. Let Θ' be the IMLL proof net such that $\Theta \to \Theta'$ by the multiplicative rewrite rule 2 w.r.t. $A^+ = A_1 \otimes A_2{}^+$ in Π and $A^- = A_1 \otimes A_2{}^-$ in η-expand(A^+, A^-). On the other hand there is an IMLL subproof net Π_1 (resp. Π_2) of Π (and also of Θ') such that Π_1 (resp. Π_2) is the maximal subproof net of Π among the subproof nets with a conclusion A_1^+ (resp. A_2^+).[1] Let Θ_1 (resp. Θ_2) be the IMLL proof net obtained by connecting Π_1 (resp. Π_2) and η-expand(A_1^+, A_1^-) (resp. η-expand(A_2^+, A_2^-)) by a Cut-link. Then both Θ_1 and Θ_2 are also an IMLL subproof net of Θ'. By applying inductive hypothesis to Θ_1 (resp. Θ_2) and Π_1 (resp. Π_2), we obtain Π_1' (resp. Π_2') from Π_1 (resp. Π_2) by some η-expansions such that $\Pi_1 \to^* \Pi_1'$ (resp. $\Pi_2 \to^* \Pi_2'$) and $\Theta_1 \to^* \Pi_1'$ (resp. $\Theta_2 \to^* \Pi_2'$). The IMLL proof net obtained from Θ' by replacing Θ_1 and Θ_2 by Π_1' and Π_2' is an IMLL proof net obtained from Π by applying the multiplicative η-expansion to some subformula occurrences of $A_1 \otimes A_2{}^+$ of Π. Let Π' be the IMLL proof net. Then $\Theta \to \Theta' \to^* \Pi'$ and $\Pi \to^* \Pi'$. □

Lemma 2 (*Weak Confluence*). *In the IMLL system we assume that $\Theta \to \Theta_1$ and $\Theta \to \Theta_2$. Then there is an IMLL proof net Θ_3 such that $\Theta_1 \to^* \Theta_3$ and $\Theta_2 \to^* \Theta_3$.*

Proof. The problematic cases are four critical pairs in Fig. 13. Let Θ_1 be the left contractum in the pairs and Θ_2 be the right contractum. Then we let Θ_1' be the IMLL proof net obtained from Θ_1 by applying the multiplicative η-expansion to Θ_1 until there are no any ID-links with non-atomic conclusions whose premises are subformulas of $A \otimes B^{pl}$ or $A \parr B^{\overline{pl}}$, where pl is an element of $\{0, 1\}$ and $\overline{pl}$ is the negation of pl. Note that $\Theta_1 \to^* \Theta_1'$. Next we apply Lemma 1 to Θ_1'. Then we can find Θ_3 such that $\Theta_2 \to^* \Theta_3$. Hence $\Theta_1' \to^* \Theta_3$. □

[1] Such a maximal subproof net is called "empire" in the literature (see [7]).

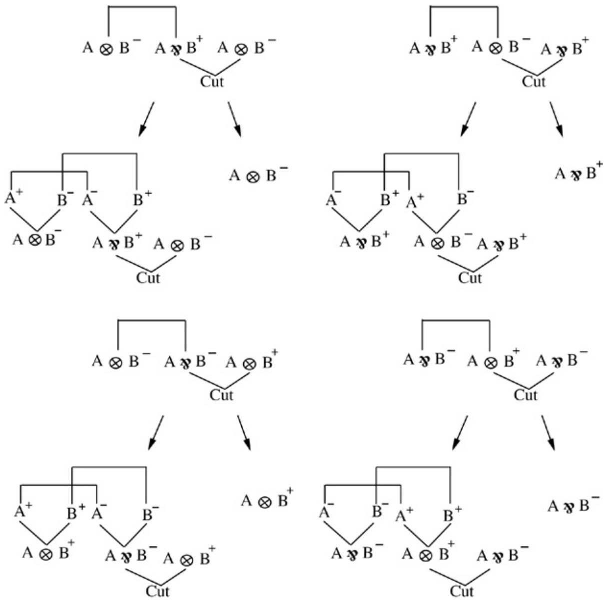


Fig. 13. All the critical pairs.

Proposition 3 (*Confluence*). *The IMLL system is confluent.*

Proof. From Proposition 2 and Lemma 2 by Newman's Lemma. □

2.2. An equality on IMLL proof nets

In this section, we define an equality on IMLL proof nets and give a characterization of the equality by *extended main paths* on normal IMLL proof nets. The equality is defined by isomorphisms on labeled directed graphs. First we start from the definition of labeled directed graphs.

Definition 11 (*Labeled Directed Graphs*). Let $\mathcal{A}$ and $\mathcal{B}$ be sets. A labeled directed graph with labels $\mathcal{A}$ and $\mathcal{B}$ is a tuple $\langle V, E, \ell_V, \ell_E \rangle$ satisfying the following conditions:

(1) V is a set;
(2) E is a set with two functions $\mathrm{src} : E \to V$ and $\mathrm{tgt} : E \to V$;
(3) ℓ_V is a function from V to $\mathcal{A}$;
(4) ℓ_E is a function from E to $\mathcal{B}$.

In the following, we suppose $\mathcal{A}$ denotes the set of IMLL formulas and $\mathcal{B}$ is $\{\mathbf{L}, \mathbf{R}, \mathbf{ID}, \mathbf{Cut}\}$. Note that $\mathcal{A}$ is *not* the set of IMLL formula *occurrences*.

Definition 12 (*Graph Isomorphisms on Labeled Directed Graphs*). Let $G_1 = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1} \rangle$ and $G_2 = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2} \rangle$ be labeled directed graphs. Then a graph homomorphism from G_1 to G_2 is a pair $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2 \rangle$ satisfying the following conditions:

(1) for any $e \in E_1$, $h_V(\mathrm{src}(e)) = \mathrm{src}(h_E(e))$ and $h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_E(e))$;
(2) for any $v \in V_1$, $\ell_{V_1}(v) = \ell_{V_2}(h_V(v))$;
(3) for any $e \in E_1$, $\ell_{E_1}(e) = \ell_{E_2}(h_E(e))$.

The graph homomorphism $\langle h_V, h_E \rangle$ is a graph isomorphism if $h_V : V_1 \to V_2$ and $h_E : E_1 \to E_2$ are both bijections.

Next, we define a translation from IMLL proof nets to labeled directed graphs. The translated graphs are basically Lamarche's essential nets, a variant of proof nets restricted to Intuitionistic Linear Logic ([10]; see also [12]).

Definition 13. Let $\Theta = \langle \mathcal{F}, \mathcal{L} \rangle$ be an IMLL proof structure. A labeled directed graph $G(\Theta) = \langle V, E, \ell_V : V \to \mathcal{A}, \ell_E : E \to \{\mathbf{L}, \mathbf{R}, \mathbf{ID}, \mathbf{Cut}\} \rangle$ is defined from Θ in the following way:

(1) $V = \{i \mid \langle A, i\rangle \in \mathcal{F}\}$ and $\ell_V = \{\langle i, A\rangle \mid \langle A, i\rangle \in \mathcal{F}\}$; since in Θ, each formula occurrence has a unique index, we can easily see that V is set-theoretically isomorphic to $\mathcal{F}$ and ℓ_V is well-defined.
(2) E and ℓ_E is the least set satisfying the following conditions:
   - If $L \in \mathcal{L}$ is an ID-link occurrence with conclusions $\langle A^+, i\rangle$ and $\langle A^-, j\rangle$, then there is an edge $e \in E$ such that $\mathrm{src}(e) = i$ and $\mathrm{tgt}(e) = j$ and $\langle e, \mathbf{ID}\rangle \in \ell_E$;
   - If $L \in \mathcal{L}$ is a Cut-link occurrence with conclusions $\langle A^+, i\rangle$ and $\langle A^-, j\rangle$, then there is an edge $e \in E$ such that $\mathrm{src}(e) = j$, $\mathrm{tgt}(e) = i$, and $\langle e, \mathbf{Cut}\rangle \in \ell_E$;
   - If $L \in \mathcal{L}$ is a $\otimes^-$-link occurrence with the form $\frac{\langle A^+,i\rangle \quad \langle B^-,j\rangle}{\langle A\otimes B^-,k\rangle}$, then there are two edges $e_1 \in E$ and $e_2 \in E$ such that $\mathrm{src}(e_1) = k$, $\mathrm{tgt}(e_1) = i$, $\mathrm{src}(e_2) = j$, $\mathrm{tgt}(e_2) = k$, $\langle e_1, \mathbf{L}\rangle \in \ell_E$, and $\langle e_2, \mathbf{R}\rangle \in \ell_E$;
   - If $L \in \mathcal{L}$ is a $\invamp^+$-link occurrence with the form $\frac{\langle A^-,i\rangle \quad \langle B^+,j\rangle}{\langle A\invamp B^+,k\rangle}$, then there are two edges $e_1 \in E$ and $e_2 \in E$ such that $\mathrm{src}(e_1) = i$, $\mathrm{tgt}(e_1) = k$, $\mathrm{src}(e_2) = k$, $\mathrm{tgt}(e_2) = j$, $\langle e_1, \mathbf{L}\rangle \in \ell_E$, and $\langle e_2, \mathbf{R}\rangle \in \ell_E$;
   - If $L \in \mathcal{L}$ is a $\otimes^+$-link occurrence with the form $\frac{\langle A^+,i\rangle \quad \langle B^+,j\rangle}{\langle A\otimes B^+,k\rangle}$, then there are two edges $e_1 \in E$ and $e_2 \in E$ such that $\mathrm{src}(e_1) = k$, $\mathrm{tgt}(e_1) = i$, $\mathrm{src}(e_2) = k$, $\mathrm{tgt}(e_2) = j$, $\langle e_1, \mathbf{L}\rangle \in \ell_E$, and $\langle e_2, \mathbf{R}\rangle \in \ell_E$;
   - If $L \in \mathcal{L}$ is a $\invamp^-$-link occurrence with the form $\frac{\langle A^-,i\rangle \quad \langle B^-,j\rangle}{\langle A\invamp B^-,k\rangle}$, then there are two edges $e_1 \in E$ and $e_2 \in E$ such that $\mathrm{src}(e_1) = i$, $\mathrm{tgt}(e_1) = k$, $\mathrm{src}(e_2) = j$, $\mathrm{src}(e_2) = k$, $\langle e_1, \mathbf{L}\rangle \in \ell_E$, and $\langle e_2, \mathbf{R}\rangle \in \ell_E$.

Proposition 4. *Let Θ be an IMLL proof structure. For any nodes v_1, v_2 in $G(\Theta)$, if an edge e in $G(\Theta)$ such that* $\mathrm{src}(e) = v_1$ *and* $\mathrm{tgt}(e) = v_2$*, then such e is unique.*

Proof. We suppose that another edge e' such that $\mathrm{src}(e') = v_1$ and $\mathrm{tgt}(e') = v_2$.

(1) The case where e is generated from an ID-link L:
   Then e' must be generated from an ID-link L', which is different from L. But it is impossible because the formula occurrences corresponding to v_1 and v_2 cannot be the conclusions of two different links L and L'.
(2) The case where e is generated from a Cut-link L:
   Then e' must be generated from a Cut-link L', which is different from L. The formula occurrences corresponding to v_1 and v_2 cannot be the premises of two different links L and L'.
(3) The case where e is generated from a $\otimes$-link or a $\invamp$-link L:
   Then e' must be generated from a link L', which has the same kind as L and is different from L. Then the formula occurrence corresponding to either v_1 or v_2 must be the conclusion of both L_1 and L_2. Without loss of generality, we can assume v_1 is the node. But is it impossible because the formula occurrence corresponding to v_1 cannot be the conclusion of two different links L_1 and L_2. □

As a consequence of the proposition above, when $\mathrm{src}(e) = i$ and $\mathrm{tgt}(e) = j$ in $G(\Theta)$, we write $e = \langle i, j\rangle$ without any mention.

Definition 14 (*The Equality = on IMLL Proof Nets*). Let Θ_1 and Θ_2 be IMLL proof nets. From the results of the previous subsection both Θ_1 and Θ_2 have the unique normal form respectively. Let the normal forms be Θ_1' and Θ_2' respectively. Θ_1 is equal to Θ_2 (denoted by $\Theta_1 = \Theta_2$), if there is an isomorphism from $G(\Theta_1')$ to $G(\Theta_2')$.

The following proposition is easy to prove.

Proposition 5. *The equality = on IMLL proof nets is reflexive, symmetric, and transitive, i.e., an equivalence relation.*

We prove two propositions, which turn out to be useful later.

Proposition 6. *Let Θ_1 and Θ_2 be IMLL proof nets. We assume that $G(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$, and $G(\Theta_2) = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle$. Moreover we assume $\Theta_1 = \Theta_2$, i.e., there is a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$. Then for any $e \in E_1$, $h_E(e) = \langle h_V(\mathrm{src}(e)), h_V(\mathrm{tgt}(e))\rangle$.*

Proof. Since $\langle h_V, h_E\rangle$ is a graph isomorphism, $\mathrm{src}(h_E(e)) = h_V(\mathrm{src}(e))$ and $\mathrm{tgt}(h_E(e)) = h_V(\mathrm{tgt}(e))$. Since an edge $e' \in E_2$ such that $\mathrm{src}(e') = h_V(\mathrm{src}(e))$ and $\mathrm{tgt}(e') = h_V(\mathrm{tgt}(e))$ is unique by Proposition 4, so $h_E(e) = \langle h_V(\mathrm{src}(e)), h_V(\mathrm{tgt}(e))\rangle$. □

Proposition 7. *We make the same assumptions as that of Proposition 6. Let $e \in E_1$ be generated from a link L in Θ_1 and $h_E(e) \in E_2$ be generated from a link L' in Θ_2. Then*

(1) *L is an ID-link iff L' is an ID-link.*
(2) *L is a Cut-link iff L' is a Cut-link.*
(3) *L is a $\otimes^{pl}$-link iff L' is a $\otimes^{pl}$-link.*
(4) *L is a $\invamp^{pl}$-link iff L' is a $\invamp^{pl}$-link.*

Proof. (1) ID-link:
We assume L is an ID-link. Then since $\ell_{E_2}(h_E(e)) = \ell_{E_1}(e) = \mathbf{ID}$. L' must be an ID-link. The reverse direction also holds since $\ell_{E_1}(e) = \ell_{E_2}(h_E(e)) = \mathbf{ID}$.

(2) Cut-link:
Similar to the case (1).

(3) $\otimes^{pl}$-link:
  (a) We assume L is a $\otimes^-$-link. We consider the case where $\ell_{E_1}(e) = \mathbf{L}$. Then $\ell_{E_2}(h_E(e)) = \ell_{E_1}(e) = \mathbf{L}$, $h_E(e) = \langle \mathrm{src}(h_E(e)), \mathrm{tgt}(h_E(e))\rangle$, $\ell_{V_2}(\mathrm{src}(h_E(e))) = \ell_{V_2}(h_V(\mathrm{src}(e))) = \ell_{V_1}(\mathrm{src}(e)) = A \otimes B^-$ and $\ell_{V_2}(\mathrm{tgt}(h_E(e))) = \ell_{V_2}(h_V(\mathrm{tgt}(e))) = \ell_{V_1}(\mathrm{tgt}(e)) = A^+$. So L' must be a $\otimes^-$-link. The case where $\ell_{E_1}(e) = \mathbf{R}$ is similar. Conversely, we assume L' is a $\otimes^-$-link. We consider the case where $\ell_{E_2}(h_E(e)) = \mathbf{L}$. Then $\ell_{E_1}(e) = \ell_{E_2}(h_E(e)) = \mathbf{L}$, $e = \langle \mathrm{src}(e), \mathrm{tgt}(e)\rangle$, $\ell_{V_1}(\mathrm{src}(e)) = \ell_{V_2}(h_V(\mathrm{src}(e))) = \ell_{V_2}(\mathrm{src}(h_E(e))) = A \otimes B^-$ and $\ell_{V_1}(\mathrm{tgt}(e)) = \ell_{V_2}(h_V(\mathrm{tgt}(e))) = \ell_{V_2}(\mathrm{tgt}(h_E(e))) = A^+$. So L must be a $\otimes^-$-link. The case where $\ell_{E_2}(h_E(e)) = \mathbf{R}$ is similar.
  (b) To prove that L is a $\otimes^+$-link iff L' is a $\otimes^+$-link is similar to the case immediately above.

(4) $\invamp^{pl}$-link:
Similar to case (3). □

Definition 15 (*Main Paths*). Let Θ be an IMLL proof net. A main path of Θ is a path $f_1, f_2, \ldots, f_n$ (where $\mathrm{src}(f_1)$ is the starting point of the path and $\mathrm{tgt}(f_n)$ the ending point) of the directed labeled graph $G(\Theta) = \langle V, E, \ell_V, \ell_E\rangle$ satisfying the following conditions:

(1) $\ell_V(\mathrm{src}(f_1))$ is the conclusion of Θ;
(2) $\ell_V(\mathrm{tgt}(f_n))$ is the left premise of a $\invamp^+$-link or a negative conclusion of Θ;
(3) for any i $(1 \le i \le n)$, if f_i is generated from a $\otimes^-$-link, then $\ell_E(f_i) \neq \mathbf{L}$.

Definition 16. An IMLL proof net Θ is closed if Θ has exactly one conclusion.

In general, an IMLL proof net Θ has several main paths. For example, Fig. 14 shows a closed IMLL proof net of $p \multimap (p \otimes p) \multimap ((p \multimap p \otimes p) \otimes (p \otimes p))^+$, where we give abbreviations to some formula occurrences. There are exactly four main paths in the IMLL proof net:

(1) $A^+ \xrightarrow{\mathbf{R}} A_1^+ \xrightarrow{\mathbf{R}} A_2^+ \xrightarrow{\mathbf{L}} A_3^+ \xrightarrow{\mathbf{R}} p \otimes p^+ \xrightarrow{\mathbf{L}} p^+ \xrightarrow{\mathbf{ID}} p^-$
(2) $A^+ \xrightarrow{\mathbf{R}} A_1^+ \xrightarrow{\mathbf{R}} A_2^+ \xrightarrow{\mathbf{L}} A_3^+ \xrightarrow{\mathbf{R}} p \otimes p^+ \xrightarrow{\mathbf{R}} p^+ \xrightarrow{\mathbf{ID}} p^- \xrightarrow{\mathbf{R}} p \otimes p^-$
(3) $A^+ \xrightarrow{\mathbf{R}} A_1^+ \xrightarrow{\mathbf{R}} A_2^+ \xrightarrow{\mathbf{R}} p \otimes p^+ \xrightarrow{\mathbf{L}} p^+ \xrightarrow{\mathbf{ID}} p^- \xrightarrow{\mathbf{L}} p \otimes p^-$
(4) $A^+ \xrightarrow{\mathbf{R}} A_1^+ \xrightarrow{\mathbf{R}} A_2^+ \xrightarrow{\mathbf{R}} p \otimes p^+ \xrightarrow{\mathbf{R}} p^+ \xrightarrow{\mathbf{ID}} p^-$.

But note that if Θ is an IIMLL proof net, then Θ has exactly one main path.

Definition 17 (*Direct Subproof Nets*). Let $\Theta = \langle \mathcal{F}, \mathcal{L}\rangle$ be an IMLL proof net and $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{L}' \subseteq \mathcal{L}$. Then $\Theta' = \langle \mathcal{F}', \mathcal{L}'\rangle$ is a subproof net of Θ if there is an IMLL proof net Θ'' such that $\Theta' = \Theta''$. A subproof net Θ' of Θ is a direct subproof net of Θ if the positive conclusion of Θ' is the left premise of a $\otimes^-$-link on a main path of Θ.

Note that there is no direct subproof net of the IMLL proof net of Fig. 14.

Definition 18. Let Θ be an IMLL proof net. A set of subproof nets of Θ $\mathbf{DSP}_\Theta$ is inductively as follows:

(1) $\Theta \in \mathbf{DSP}_\Theta$;
(2) If $\Theta' \in \mathbf{DSP}_\Theta$ and Θ'' is a direct subproof net of Θ', then $\Theta'' \in \mathbf{DSP}_\Theta$.

That is, $\mathbf{DSP}_\Theta$ is the reflexive transitive closure of the direct subproof net relation on Θ.

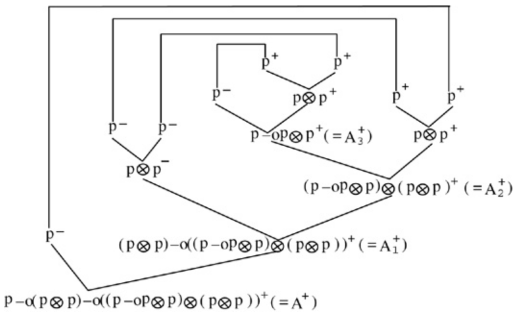


Fig. 14. A closed IMLL proof net of $p \multimap (p \otimes p) \multimap ((p \multimap p \otimes p) \otimes (p \otimes p))^+$.

Definition 19 (*Extended Main Paths*). Let Θ be an IMLL proof net, $\Theta' \in \mathbf{DSP}_\Theta$, and $s = e_1, \ldots, e_n$ be a main path of Θ'. The extended main path of s is the path on $G(\Theta)$ obtained from s by adding (a) the edge e_0 in which $\mathrm{src}(e_0)$ is the conclusion of a $\otimes^-$-link and $\ell_E(e_0) = \mathbf{L}$ to the starting point of s, if any, and (b) the edge e_{n+1} in which $\mathrm{tgt}(e_{n+1})$ is the conclusion of a $⅋^+$-link and $\ell_E(e_{n+1}) = \mathbf{L}$ to the ending point of s, if any. Moreover we define the set $\mathbf{EMP}_\Theta$ by

$$\mathbf{EMP}_\Theta = \bigcup_{\Theta' \in \mathbf{DSP}_\Theta} \{s \mid s \text{ is an extended main path of } \Theta'\}.$$

The following proposition states that we can extend each path $s \in \mathbf{EMP}_\Theta$ to a path that starts at the positive conclusion of Θ.

Proposition 8. *Let Θ be an IMLL proof net. We assume that $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_\Theta$. Then for each i ($1 \le i \le m$), there is a path in $G(\Theta)$ from the positive conclusion of Θ to* $\mathrm{tgt}(e_i)$ *including the subpath $e_0, \ldots, e_i$.*

Proof. Let Θ' and Θ'' be elements of $\mathbf{DSP}_\Theta$. Then we write $\Theta'' < \Theta'$ when Θ'' is a direct subproof net of Θ'. If $\Theta_m < \Theta_{m-1} < \cdots < \Theta_1 < \Theta$, then we say that Θ_m has the height m in Θ (we write $\mathrm{hght}_\Theta(\Theta_m) = m$). Moreover we say Θ has the height 0 (so we write $\mathrm{hght}_\Theta(\Theta) = 0$). Let $f_1, \ldots, f_n$ be the main path corresponding to $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_\Theta$. Then we define $\mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m)$ to be

$$\max\{\mathrm{hght}_\Theta(\Theta') \mid \Theta' \in \mathbf{DSP}_\Theta \wedge G(\Theta') = \langle V', E', \ell_{V'}, \ell_{E'}\rangle \wedge \{f_1, \ldots, f_n\} \subseteq E'\}.$$

We prove the proposition by the induction on $\mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m)$.

(1) The case where $\mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m) = 0$:
Since $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_\Theta$, $\mathrm{src}(e_0)$ must be the conclusion of Θ. Then we can easily see $e_0, e_1, \ldots, e_m$ is a path from the conclusion of Θ to $\mathrm{tgt}(e_i)$ for each i ($1 \le i \le m$).

(2) The case where $\mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m) > 0$:
Let $f_1, \ldots, f_n$ be the main path corresponding to $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_\Theta$. Let Θ' be the element of $\mathbf{DSP}_\Theta$ such that $\mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m) = \mathrm{hght}_\Theta(\Theta')$ and when we write $G(\Theta') = \langle V', E', \ell_{V'}, \ell_{E'}\rangle$, $\{f_1, \ldots, f_n\} \subseteq E'$. Since $\mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m) > 0$, Θ' is not Θ. So, by the definition of extended main paths, $\mathrm{src}(e_0)$ is the conclusion of a $\otimes^-$-link L in Θ, but not belonging to Θ'. Then there is an edge e' in $G(\Theta) = \langle V, E, \ell_V, \ell_E\rangle$ such that $\mathrm{tgt}(e') = \mathrm{src}(e_0)$, $\ell_E(e') = \mathbf{R}$, and $\mathrm{src}(e')$ is the right premise of L. On the other hand, since $\Theta' \in \mathbf{DSP}_\Theta$ and $\Theta' \neq \Theta$, there is a $\Theta'' \in \mathbf{DSP}_\Theta$ such that Θ' is a direct subproof net of Θ''. Then, when we write $G(\Theta'') = \langle V'', E'', \ell_{V''}, \ell_{E''}\rangle$, $\mathrm{tgt}(e') = \mathrm{src}(e_0) \in V''$ and e' is on a main path $f''_1, \ldots, f''_{n''}$ of Θ'' by the definition of direct subproof nets. Let $e''_0, \ldots, e''_{m''}$ be the extended main path corresponding to $f''_1, \ldots, f''_{n''}$. Then $\mathrm{hght}_\Theta(e''_0, \ldots, e''_{m''}) < \mathrm{hght}_\Theta(e_0, e_1, \ldots, e_m)$ because $\{f''_1, \ldots, f''_{n''}\} \subseteq E''$ but not $\{f''_1, \ldots, f''_{n''}\} \subseteq E'$, and $\Theta' < \Theta''$. Hence by inductive hypothesis there is a path from the conclusion of Θ to $\mathrm{tgt}(e')$ including e' because $e' \in \{f''_1, \ldots, f''_{n''}\}$. Let the path be $g_1, \ldots, g_k, e'$. Then $g_1, \ldots, g_k, e', e_0, \ldots, e_{i-1}, e_i$ is a path from the conclusion of Θ to $\mathrm{tgt}(e_i)$ for each i ($0 \le i \le m$). $\square$

Conversely, the following proposition also holds.

Proposition 9. *Let Θ be an IMLL proof net, $G(\Theta) = \langle V, E, \ell_V, \ell_E\rangle$, and $e \in E$. Then if there is a path $g_1, \ldots, g_k, e$ ($k \geq 0$) in $G(\Theta)$, from the positive conclusion of Θ to $\mathrm{tgt}(e)$, then there is an element $e_0, \ldots, e_m$ of $\mathbf{EMP}_\Theta$ such that $e \in \{e_0, \ldots, e_m\}$.*

Proof. From the assumption, there is a path including the edge e with the least length $\min(e)(\geq 1)$ from the positive conclusion of Θ to $\mathrm{tgt}(e)$. Let such a path be $g'_1, \ldots, g'_{\min(e)-1}, e$. Then we proceed by induction on least lengths of such paths.

(1) The case where $\min(e) = 1$:
Then $\mathrm{src}(e)$ must be the positive conclusion of Θ. Moreover the link corresponding to the edge e is an ID-link, a $\otimes^+$-link, or $⅋^+$-link. From the definition of main paths, e must belong to a main path $t = f_1, \ldots, f_n$ of Θ (in fact $e = f_1$). When we let the extended main path of t be $e_0, \ldots, e_m$, then $e \in \{f_1, \ldots, f_n\} \subseteq \{e_0, \ldots, e_m\}$.
(2) The case where $\min(e) > 1$:
   (a) The case where $\ell_E(e) = \mathbf{ID}$:
   We consider $g'_1, \ldots, g'_{\min(e)-2}, g'_{\min(e)-1}$ and $g'_{\min(e)-1} \in E$. Then by inductive hypothesis, there is an element $e_0, \ldots, e_m$ of $\mathbf{EMP}_\Theta$ such that $g'_{\min(e)-1} \in \{e_0, \ldots, e_m\}$. Since $\mathrm{tgt}(g'_{\min(e)-1})$ is the positive conclusion of the ID-link corresponding to e, from the definition of main paths and extended main paths $e \in \{e_0, \ldots, e_m\}$.
   (b) The case where the link corresponding to e is a $\otimes^-$-link and $\ell_E(e) = \mathbf{L}$: We consider $g'_1, \ldots, g'_{\min(e)-2}, g'_{\min(e)-1}$ and $g'_{\min(e)-1} \in E$. Then by inductive hypothesis, there is an element $e_0, \ldots, e_m$ of $\mathbf{EMP}_\Theta$ such that $g'_{\min(e)-1} \in \{e_0, \ldots, e_m\}$. Then both e and $g'_{\min(e)-1}$ are generated from the same $\otimes^-$-link and $\ell_E(g'_{\min(e)-1}) = \mathbf{R}$. Then there is an element Θ' of $\mathbf{DSP}_\Theta$ such that the positive conclusion of Θ' is $\mathrm{tgt}(e)$. Then there is a main path of Θ' $f'_1, \ldots, f'_{n'}$. When we let the extended path of $f'_1, \ldots, f'_{n'}$ be $e'_1, \ldots, e'_{m'}$. Then $e \in \{e'_1, \ldots, e'_m\}$.
   (c) The case where $\ell_E(e) = \mathbf{Cut}$:
   Similar to the case (2a).
   (d) The case where the link corresponding to e is a $\otimes^+$-link, a $⅋^-$-link, or a $⅋^+$-link and $\ell_E(e) = \mathbf{L}$ or $\ell_E(e) = \mathbf{R}$:
   Similar to the case (2a).
   (e) The case where the link corresponding to e is a $\otimes^-$-link and $\ell_E(e) = \mathbf{R}$: Similar to the case (2a). □

Next, we give a characterization of normal IMLL proof nets in terms of $\mathbf{EMP}_\Theta$.

Proposition 10. *Let Θ_1 and Θ_2 be normal IMLL proof nets. Let $G(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$ and $G(\Theta_2) = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle$. Then $\Theta_1 = \Theta_2$ iff*

(1) *there is a bijection $h_V : V_1 \to V_2$ such that for each $v \in V_1$, $\ell_{V_1}(v) = \ell_{V_2}(h_V(v))$;*
(2) *there is a bijection $h_{\mathbf{EMP}} : \mathbf{EMP}_{\Theta_1} \to \mathbf{EMP}_{\Theta_2}$ such that for each $s = e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$, when we write $h_{\mathbf{EMP}}(s) = e'_0, e'_1, \ldots, e'_{m'}$, then $m = m'$ and for each i $(0 \leq i \leq m)$,*
   (a) $\ell_{E_1}(e_i) = \ell_{E_2}(e'_i)$;
   (b) $h_V(\mathrm{src}(e_i)) = \mathrm{src}(e'_i)$;
   (c) $h_V(\mathrm{tgt}(e_i)) = \mathrm{tgt}(e'_i)$.

Proof. • *only-if part*:
Because $\Theta_1 = \Theta_2$, by definition we have a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$. The condition (1) is a part of the graph isomorphism.
We define

$$h_{\mathbf{EMP}} = \{\langle (e_0, e_1, \ldots, e_m), (h_E(e_0), h_E(e_1), \ldots, h_E(e_m))\rangle \mid e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}\}.$$

It is obvious that $h_{\mathbf{EMP}}$ is a function with domain $\mathbf{EMP}_{\Theta_1}$ (but at the current point, it is unclear whether or not the codomain is $\mathbf{EMP}_{\Theta_2}$).
Let $e'_i = h_E(e_i)$ for i $(0 \leq i \leq m)$.
• $e'_0, e'_1, \ldots, e'_m$ is a path of $G(\Theta_2)$.
Since $e_0, e_1, \ldots, e_m$ is a path in $G(\Theta_1)$, for each i $(0 \leq i \leq m)$, $\mathrm{tgt}(e_i) = \mathrm{src}(e_{i+1})$. Then $\mathrm{tgt}(e'_i) = \mathrm{tgt}(h_E(e_i)) = h_V(\mathrm{tgt}(e_i)) = h_V(\mathrm{src}(e_{i+1})) = \mathrm{src}(h_E(e_{i+1})) = \mathrm{src}(e'_{i+1})$.

- $e'_0, e'_1, \ldots, e'_m$ is in $\mathbf{EMP}_{\Theta_2}$.

  (1) The case where $\mathrm{src}(e_0)$ is the conclusion of Θ_1:
  Since $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$, $e_0, e_1, \ldots, e_m$ must be an extended main path of Θ_1 itself. Since $e'_0 = h_E(e_0)$, $\mathrm{src}(e'_0)$ must be the conclusion of Θ_2. Moreover since $e'_i = h_E(e_i)$ for each i $(1 \le i \le m)$, the type of the link corresponding to e'_i has the same as that of e_i from Proposition 7 for each i $(1 \le i \le m)$. Hence from the definition of extended main paths, $e'_0, e'_1, \ldots, e'_m$ is an extended main path of Θ_2 itself. So $e'_0, e'_1, \ldots, e'_m \in \mathbf{EMP}_{\Theta_2}$, too.

  (2) The case where $\mathrm{src}(e_0)$ is not the conclusion of Θ_1:
  Since $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$, $\mathrm{src}(e_0)$ must be the left premise of a $\otimes^-$-link in Θ_1. Since $e'_0 = h_E(e_0)$, from Proposition 7 $\mathrm{src}(e'_0)$ is also the left premise of a $\otimes^-$-link in Θ_2. On the other hand, since $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$, there is a path $f_1, \ldots, f_{n-1}, f_n$ in $G(\Theta_1)$ from the positive conclusion of Θ_1 to $\mathrm{tgt}(e_0)$ including the subpath e_0 (hence $f_n = e_0$) from Proposition 8. Then $h_E(f_1), \ldots, h_E(f_{n-1})$ is a path from the positive conclusion of Θ_2 to $\mathrm{tgt}(h_E(f_{n-1})) = h_V(\mathrm{tgt}(f_{n-1})) = h_V(\mathrm{src}(e_0)) = \mathrm{src}(h_E(e_0)) = \mathrm{src}(e'_0)$. Then from Proposition 9, there is an element $e''_0, \ldots, e''_k$ in $\mathbf{EMP}_{\Theta_2}$ such that $h_E(f_{n-1})$ in $\{e''_0, \ldots, e''_k\}$. Since $\mathrm{src}(e'_0) = \mathrm{tgt}(h_E(f_{n-1}))$, e'_0 and $h_E(f_{n-1})$ are generated from the same $\otimes^-$-link in Θ_2. Then since $e'_i = h_E(e_i)$ for each i $(1 \le i \le m)$, the type of the link corresponding to e'_i has the same as that of e_i from Proposition 7 for each i $(1 \le i \le m)$. Hence from the definition of extended main paths and $e_0, e_1, \ldots, e_m$ in $\mathbf{EMP}_{\Theta_1}$, $e'_0, e'_1, \ldots, e'_m$ must be in $\mathbf{EMP}_{\Theta_2}$, too.

Hence we conclude that $h_{\mathbf{EMP}} : \mathbf{EMP}_{\Theta_1} \to \mathbf{EMP}_{\Theta_2}$.
On the other hand, when given a path $e'_0, e'_1, \ldots, e'_{m'}$ in $\mathbf{EMP}_{\Theta_2}$, $h_E^{-1}(e'_0), h_E^{-1}(e'_1), \ldots, h_E^{-1}(e'_{m'}) \in \mathbf{EMP}_{\Theta_1}$ is proved similarly above. So, $h_{\mathbf{EMP}}$ is surjective. Moreover the injectivity of $h_{\mathbf{EMP}}$ is derived from the injectivity of h_E.
Let $V_{e_0,e_1,\ldots,e_m}$ be $\{\mathrm{src}(e_i) \mid 0 \le i \le m\} \cup \{\mathrm{tgt}(e_i) \mid 0 \le i \le m\}$.
Then, $h_V(V_{e_0,e_1,\ldots,e_m}) = \{\mathrm{src}(e'_i) \mid 0 \le i \le m\} \cup \{\mathrm{tgt}(e'_i) \mid 0 \le i \le m\}$, because

$$\begin{aligned} h_V(V_{e_0,e_1,\ldots,e_m}) &= \{h_V(\mathrm{src}(e_i)) \mid 0 \le i \le m\} \cup \{h_V(\mathrm{tgt}(e_i)) \mid 0 \le i \le m\} \\ &= \{\mathrm{src}(h_E(e_i)) \mid 0 \le i \le m\} \cup \{\mathrm{tgt}(h_E(e_i)) \mid 0 \le i \le m\} \\ &= \{\mathrm{src}(e'_i) \mid 0 \le i \le m\} \cup \{\mathrm{tgt}(e'_i)) \mid 0 \le i \le m\}. \end{aligned}$$

So, $\langle h_V|_{V_{e_0,e_1,\ldots,e_m}}, h_E|_{\{e_0,e_1,\ldots,e_m\}}\rangle$ is a graph isomorphism from path $e_0, e_1, \ldots, e_m$ to $e'_0, e'_1, \ldots, e'_m$. Hence we can conclude that the conditions (a), (b), and (c) hold for $\{e_0, e_1, \ldots, e_m\}$ and $\{e'_0, e'_1, \ldots, e'_m\}$. Finally we conclude that $h_{\mathbf{EMP}} : \mathbf{EMP}_{\Theta_1} \to \mathbf{EMP}_{\Theta_2}$ is a bijection satisfying (a), (b), and (c).

- *if part*:
By the assumption (2), for each $s = e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$, there is a unique $h_{\mathbf{EMP}}(s) = e'_0, e'_1, \ldots, e'_m \in \mathbf{EMP}_{\Theta_2}$ such that for each i $(0 \le i \le m)$, the conditions (a), (b), and (c) hold. Then we define $h_s : \{e_0, e_1, \ldots, e_m\} \to \{e'_0, e'_1, \ldots, e'_m\}$ by $e_i \mapsto e'_i$ $(0 \le i \le m)$. Then we define h_E by

$$h_E = \bigcup_{s \in \mathbf{EMP}_{\Theta_1}} h_s.$$

Next, we prove that h_E is a bijection from E_1 to E_2:

- $\mathrm{dom}(h_E) = E_1$:
We suppose that $\mathrm{dom}(h_E) \ne E_1$. Since $\mathrm{dom}(h_E) \subseteq E_1$, there is an $e \in E_1$ such that $e \notin \mathrm{dom}(h_E)$. Then for any $e_0, e_1, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$, $e \notin \{e_0, e_1, \ldots, e_m\}$.
Next in the IMLL proof net Θ_1, we choose a Danos–Regnier switching S as follows:
for each $⅋^+$-link L, we set $S(L) = 1$, i.e., we choose the right premise;
for each $⅋^-$-link L, we set $S(L) = 0$ or $S(L) = 1$ nondeterministically.
From Theorem 1, the undirected graph Θ_{1S} must be connected. So, in Θ_{1S}, there is a (undirected) path from the positive conclusion to $\mathrm{tgt}(e)$. Then from the way we choose the switching S we can find that, in the directed graph $G(\Theta_1)$, there is also a directed path from the positive conclusion of Θ_1 to $\mathrm{tgt}(e)$. But from Proposition 9 there is an element $e_0, \ldots, e_m$ of $\mathbf{EMP}_{\Theta_1}$ such that $e \in \{e_0, \ldots, e_m\}$. But this is a contradiction.

- h_E is a function:

  We suppose that $e \in E_1$, $\langle e, e'_1\rangle \in h_E$, and $\langle e, e'_2\rangle \in h_E$. By the definition of h_E, there are two paths s and s' in $\mathbf{EMP}_{\Theta_1}$ such that $h_s(e) = e'_1$ and $h_{s'}(e) = e'_2$. Then $\mathrm{src}(e'_1) = \mathrm{src}(h_s(e)) = h_V(\mathrm{src}(e)) = \mathrm{src}(h_{s'}(e)) = \mathrm{src}(e'_2)$ and $\mathrm{tgt}(e'_1) = \mathrm{tgt}(h_s(e)) = h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_{s'}(e)) = \mathrm{tgt}(e'_2)$. Hence e'_1 and e'_2 has the same source and the same target in $G(\Theta_2)$. From Proposition 4, $e'_1 = e'_2$.
- h_E is an injection:

  We suppose that $e' \in E_2$, $\langle e_1, e'\rangle \in h_E$, and $\langle e_2, e'\rangle \in h_E$. By the definition of h_E, there are two paths s and s' in $\mathbf{EMP}_{\Theta_1}$ such that $h_s(e_1) = e'$ and $h_{s'}(e_2) = e'$. Then $h_V(\mathrm{src}(e_1)) = \mathrm{src}(h_s(e_1)) = \mathrm{src}(e') = \mathrm{src}(h_{s'}(e_2)) = h_V(\mathrm{src}(e_2))$ and $h_V(\mathrm{tgt}(e_1)) = \mathrm{tgt}(h_s(e_1)) = \mathrm{tgt}(e') = \mathrm{tgt}(h_{s'}(e_2)) = h_V(\mathrm{tgt}(e_2))$. Then since h_V is an injection, $\mathrm{src}(e_1) = \mathrm{src}(e_2)$ and $\mathrm{tgt}(e_1) = \mathrm{tgt}(e_2)$. Hence e_1 and e_2 has the same source and the same target in $G(\Theta_1)$. From Proposition 4, $e_1 = e_2$.
- h_E is a surjection:

  We suppose that h_E is not a surjection. Then there is an edge $e' \in \mathrm{cod}(h_E) \subseteq E_2$ such that for any extended main path $e''_0, \ldots, e''_n \in \mathbf{EMP}_{\Theta_1}$, $e' \notin \{h_E(e''_0), \ldots, h_E(e''_n)\}$. Then $\{h_E(e''_0), \ldots, h_E(e''_n)\} = \{h_{e''_0,\ldots,e''_n}(e''_0), \ldots, h_{e''_0,\ldots,e''_n}(e''_n)\} = \{h_{\mathbf{EMP}}(e''_0, \ldots, e''_n)\}$ from the definition of h_E. Moreover for any $e'_0, \ldots, e'_m \in \mathbf{EMP}_{\Theta_2}$ there is an extended main path $e_0, \ldots, e_m \in \mathbf{EMP}_{\Theta_1}$ such that $e'_0, \ldots, e'_m = h_{\mathbf{EMP}}(e_0, \ldots, e_m)$ from the assumption (2). Hence, we can derive that for any $e'_0, e'_1, \ldots, e'_m \in \mathbf{EMP}_{\Theta_2}$, $e' \notin \{e'_0, e'_1, \ldots, e'_m\}$. Next in the IMLL proof net Θ_2, we choose a Danos–Regnier switching S as follows:

  for each $\parr^+$-link L, we set $S(L) = 1$, i.e., we choose the right premise;

  for each $\parr^-$-link L, we set $S(L) = 0$ or $S(L) = 1$ nondeterministically.

  From Theorem 1, the undirected graph Θ_{2S} must be connected. So, in Θ_{2S}, there is a (undirected) path from the positive conclusion to $\mathrm{tgt}(e')$. Then from the way we choose the switching S we can find that, in the directed graph $G(\Theta_2)$, there is also a directed path from the positive conclusion of Θ_2 to $\mathrm{tgt}(e')$. But from Proposition 9 there is an element $e'_0, \ldots, e'_m$ of $\mathbf{EMP}_{\Theta_2}$ such that $e' \in \{e'_0, \ldots, e'_m\}$. But this is a contradiction.

Next, we prove that $h_E : E_1 \to E_2$ satisfies the conditions (1), (2), and (3) of Definition 12.

- The condition (1):

  We assume that $e \in E_1$. Then there is a subset h_s of h_E such that $s \in \mathbf{EMP}_{\Theta_1}$ and $e \in dom(h_s)$. Then by (b) and (c) of the assumption (2) about $s \in \mathbf{EMP}_{\Theta_1}$, $h_V(\mathrm{src}(e)) = \mathrm{src}(h_s(e)) = \mathrm{src}(h_E(e))$ and $h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_s(e)) = \mathrm{tgt}(h_E(e))$.
- The condition (2):

  This is just the assumption (1).
- The condition (3):

  We assume that $e \in E_1$. Then there is a subset h_s of h_E such that $s \in \mathbf{EMP}_{\Theta_1}$ and $e \in dom(h_s)$. Then by (a) of assumption (2) about $s \in \mathbf{EMP}_{\Theta_1}$, $\ell_{E_1}(e) = \ell_{E_2}(h_s(e)) = \ell_{E_2}(h_E(e))$. $\square$

Example 1. The IIMLL proof net of Fig. 14 (let the net be Θ_1^a) and that of Fig. 15 (let the net be Θ_2^a) are two IMLL proof nets with the same conclusion. But $\Theta_1^a \neq \Theta_2^a$, because there is no extended main path in Θ_2 to corresponding to $A^+ \overset{\mathbf{R}}{\to} A_1^+ \overset{\mathbf{R}}{\to} A_2^+ \overset{\mathbf{R}}{\to} p \otimes p^+ \overset{\mathbf{L}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{L}}{\to} p \otimes p^- \overset{\mathbf{L}}{\to} A_1^+$ in Θ_1.

Example 2. As another example, Fig. 16 shows why main paths are not enough characterize the equality = of IMLL proof nets. Let Θ_3 be the left proof net of Fig. 16 and Θ_4 be the right proof net. Then the main paths of the elements of $\mathbf{DSP}_{\Theta_3}$ are the following four:

(1) $A^+ \overset{\mathbf{R}}{\to} A_1^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{R}}{\to} (p \multimap p) \multimap p^- \overset{\mathbf{R}}{\to} B^-$

(2) $p \multimap p^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{R}}{\to} p \multimap p^-$

(3) $p^+ \overset{\mathbf{ID}}{\to} p^-$

(4) $p \multimap p^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^-$.

Moreover the main paths of the elements of $\mathbf{DSP}_{\Theta_4}$ are the same as that of $\mathbf{DSP}_{\Theta_4}$. But since $\Theta_3 \neq \Theta_4$, we cannot characterize the equality = on IMLL proof nets in terms of main paths. On the other hand, $\mathbf{EMP}_{\Theta_3}$ exactly consists of the following four elements:

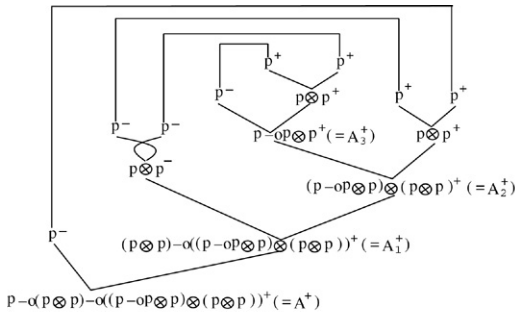


Fig. 15. Another closed IMLL proof net of $p \multimap (p \otimes p) \multimap ((p \multimap p \otimes p) \otimes (p \otimes p))^+$.

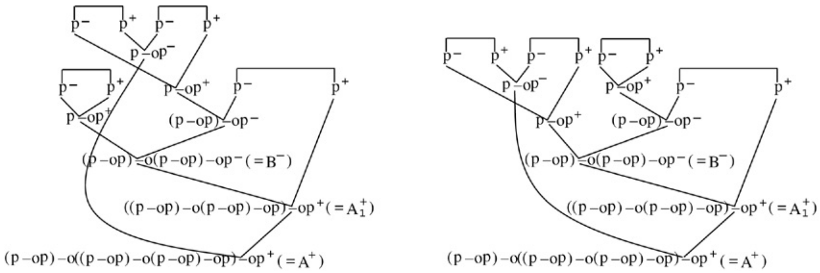


Fig. 16. An example showing why main paths are not enough.

(1) $A^+ \overset{\mathbf{R}}{\to} A_1^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{R}}{\to} (p \multimap p) \multimap p^- \overset{\mathbf{R}}{\to} B^- \overset{\mathbf{L}}{\to} A_1^+$

(2) $(p \multimap p) \multimap p^- \overset{\mathbf{L}}{\to} p \multimap p^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{R}}{\to} p \multimap p^- \overset{\mathbf{L}}{\to} A^+$

(3) $p \multimap p^- \overset{\mathbf{L}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{L}}{\to} p \multimap p^+$

(4) $B^- \overset{\mathbf{L}}{\to} p \multimap p^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{L}}{\to} p \multimap p^+$.

Moreover $\mathbf{EMP}_{\Theta_4}$ exactly consists of the following four elements:

(1) $A^+ \overset{\mathbf{R}}{\to} A_1^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{R}}{\to} (p \multimap p) \multimap p^- \overset{\mathbf{R}}{\to} B^- \overset{\mathbf{L}}{\to} A_1^+$

(2) $(p \multimap p) \multimap p^- \overset{\mathbf{L}}{\to} p \multimap p^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{L}}{\to} p \multimap p^+$

(3) $B^- \overset{\mathbf{L}}{\to} p \multimap p^+ \overset{\mathbf{R}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{R}}{\to} p \multimap p^- \overset{\mathbf{L}}{\to} p \multimap p^+$

(4) $p \multimap p^- \overset{\mathbf{L}}{\to} p^+ \overset{\mathbf{ID}}{\to} p^- \overset{\mathbf{L}}{\to} A^+$.

Then it is obvious that there is no bijection between $\mathbf{EMP}_{\Theta_3}$ and $\mathbf{EMP}_{\Theta_4}$ satisfying the condition of Proposition 10.

3. Third-order reduction on IMLL proof nets

In this section, we prove that when we are given two closed IMLL proof nets Θ_1 and Θ_2 with the same conclusion A^+ such that $\Theta_1 \neq \Theta_2$, we can find an IMLL proof net $C[]$ with two conclusions A^- and A_0^+, where A_0^+ is an IIMLL formula with order less than 4, such that $C[\Theta_1] \neq C[\Theta_2]$, where $C[\Theta_1]$ (respectively $C[\Theta_2]$) is an IMLL proof net (in fact IIMLL proof net) obtained from Θ_1 (resp. Θ_2) by connecting $C[]$ using a Cut-link. In early drafts of this paper we regarded this statement as the linear version of the third-order reduction technique of the simply typed lambda calculus [14–16]. But two referees pointed out that the most part of this reduction technique is the linear distributive law $(A \parr B) \otimes C \multimap A \parr (B \otimes C)$ (see [2]). We note that we have rewritten this section, following a suggestion of a referee such that the proof is simplified, compared to that of early drafts, and the linear distributive law has been used in [18] for a similar purpose.

We start with some definitions.

Definition 20 (*The Order of a Positive IIMLL Formula*). The order of an IIMLL formula A^+, denoted by order(A^+), is inductively as follows:

(1) If A^+ is an atomic formula p^+ then order(A^+) is 1.
(2) If A^+ is $A_1 \multimap \cdots \multimap A_n \multimap p^+$, then order($A^+$) is

$$\max\{\text{order}(A_1^+), \ldots, \text{order}(A_n^+)\} + 1.$$

We define the *order* of a closed IIMLL proof net Θ as the order of the positive conclusion.

Definition 21 (*Wrapping Nets, their Equality, and their Composition*). A wrapping net from A_1^+ to A_2^+ is a normal IMLL proof net with the positive conclusion A_2^+ and the only one negative conclusion A_1^-. Such a wrapping net is generally denoted by $C^{A_1^+}_{A_2^+}[]$. The superscript and the subscript of a wrapping net are often omitted if they are clear from textual contexts.

Let Θ be an IMLL proof net with the positive conclusion A_1^+. Then $C^{A_1^+}_{A_2^+}[\Theta]$ is the IMLL proof net obtained connecting Θ and $C^{A_1^+}_{A_2^+}[]$ using the Cut-link with the premises A_1^+ and A_1^-.

Let $C'^{A_1^+}_{A_2^+}[]$ and $C''^{A_1^+}_{A_2^+}[]$ be two wrapping nets from A_1^+ to A_2^+. Then $C'^{A_1^+}_{A_2^+}[] = C''^{A_1^+}_{A_2^+}[]$ if they are equal as IMLL proof nets, i.e., there is a graph isomorphism between $G(C'^{A_1^+}_{A_2^+}[])$ and $G(C''^{A_1^+}_{A_2^+}[])$.

Let $C^{A^+}_{B^+}[]$ and $C'^{B^+}_{C^+}[]$ be wrapping nets from A^+ to B^+ and B^+ to C^+, respectively. Then we define $C'^{B^+}_{C^+}[C^{A^+}_{B^+}[]]$ to be the IMLL proof net obtained connecting $C^{A^+}_{B^+}[]$ and $C'^{B^+}_{C^+}[]$ using the Cut-link with the premises B^+ and B^-.

Wrapping nets are an analogue to contexts in λ-calculus. But they are simpler than contexts because free variable capturing never occurs.

Lemma 3. *Let $C^{A^+}_{B^+}[]$ and $C'^{B^+}_{C^+}[]$ be wrapping nets from A^+ to B^+ and B^+ to C^+, respectively. Then there is a wrapping net $C''^{A^+}_{C^+}[]$ from A^+ to C^+ such that $C''^{A^+}_{C^+}[] = C'^{B^+}_{C^+}[C^{A^+}_{B^+}[]]$.*

Proof. We let $C''^{A^+}_{C^+}[]$ be the normal form of $C'^{B^+}_{C^+}[C^{A^+}_{B^+}[]]$. □

Definition 22 (*Pseudo IIMLL Formulas*). A positive IMLL formula A^+ is a pseudo IIMLL formula if A^+ has the form $B_1 \multimap \cdots \multimap B_n \multimap p^+$ $(n \geq 0)$, where B_i is an IMLL formula for each i $(1 \leq i \leq n)$. An IMLL proof net Θ is a pseudo IIMLL proof net if Θ has the positive conclusion of a pseudo IIMLL formula.

In the rest of this section, we reduce IMLL proof nets in the following steps:

- Step 1.
  When a given closed IMLL proof net Θ, we find a wrapping net $C[]$ which transforms Θ to a closed pseudo IIMLL proof net. Moreover we show that the transformation by $C[]$ is injective.
- Step 2.
  When a given closed pseudo IIMLL proof net Θ, we find a wrapping net $C[]$ such that $\Theta \Rightarrow_{\mathrm{LDR}_i} C[\Theta]$ for some i $(1 \leq i \leq 5)$ and $C[\Theta]$ is a closed pseudo IIMLL proof net until these transformations cannot be applied, where $\Rightarrow_{\mathrm{LDR}_i}$ for each i $(1 \leq i \leq 5)$ is a reduction relation which is an intuitionistic version of the linear distributive law. Moreover we show these transformations are injective.
- Step 3.
  If we cannot apply Step 2 to a closed pseudo proof net IIMLL Θ, then the positive conclusion A^+ of Θ must have the form $F_1 \multimap \cdots \multimap F_n \multimap p^+$ $(n \geq 1)$, where $F_i = \overbrace{p \otimes \cdots \otimes p}^{k_1} \multimap \cdots \multimap \overbrace{p \otimes \cdots \otimes p}^{k_{\ell_i}} \multimap p$ $(1 \leq i \leq n, \ell_i \geq 1, 1 \leq j \leq \ell_i, k_j \geq 1)$ or $F_i = \overbrace{p \otimes \cdots \otimes p}^{m_i}$ $(m_i \geq 1)$. Then we say that A^+ is an **essentially third-order formula**. Then we find a wrapping net $C^{A^+}_{A_0^+}[]$ such that A_0^+ is an IIMLL formula with order less than 4. Moreover we show that the transformation by $C^{A^+}_{A_0^+}[]$ is injective. This transformation is the well-known currying/uncurrying isomorphism.

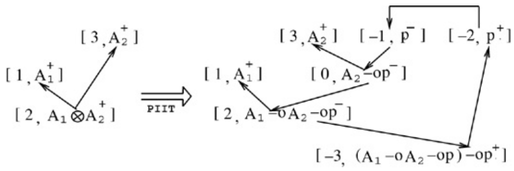


Fig. 17. Pseudo IIMLL transformation.

3.1. Step 1: Transforming IMLL proof nets into pseudo IIMLL proof nets

Definition 23 (*Pseudo IIMLL Transformation*). Let Θ_1 be a normal IMLL proof net with the positive conclusion $A_1 \otimes A_2$. Then $G(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1} \rangle$ must include the subgraph of the left side of Fig. 17, where $[i, F]$ in the figure means that $i \in V_1$ and $\ell_{V_1}(i) = F$. (More precisely, $G(\Theta_1)$ must include a graph isomorphic to the subgraph.) Then let $\langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'} \rangle$ be the labeled directed graph obtained from $G(\Theta_1)$ replacing the left side of Fig. 17 by the right side. More precisely, $\langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'} \rangle$ consists of the following data:

- $V_1' = V_1 \uplus \{0, -1, -2, -3\}$;
- $E_1' = (E_1 - \{\langle 2, 3 \rangle\}) \uplus (\{\langle 0, 2 \rangle, \langle -1, 0 \rangle, \langle 0, 3 \rangle, \langle -2, -1 \rangle, \langle -3, 2 \rangle, \langle -2, -3 \rangle\})$;
- $\ell_{V_1'}(v) = \begin{cases} \ell_{V_1}(v) & \text{if } v \in V_1 - \{2\} \\ A_1 \multimap A_2 \multimap p^- & \text{if } v = 2 \\ A_2 \multimap p^- & \text{if } v = 0 \\ p^- & \text{if } v = -1 \\ p^+ & \text{if } v = -2 \\ (A_1 \multimap A_2 \multimap p) \multimap p^+ & \text{if } v = -3 \end{cases}$
- $\ell_{E_1'}(e) = \begin{cases} \ell_{E_1}(e) & \text{if } e \in E_1 - \{\langle 2, 3 \rangle\} \\ \mathbf{R} & \text{if } e = \langle 0, 2 \rangle \\ \mathbf{L} & \text{if } e = \langle 0, 3 \rangle \\ \mathbf{R} & \text{if } e = \langle -1, 0 \rangle \\ \mathbf{ID} & \text{if } e = \langle -2, -1 \rangle \\ \mathbf{L} & \text{if } e = \langle 2, -3 \rangle \\ \mathbf{R} & \text{if } e = \langle -3, -2 \rangle. \end{cases}$

Lemma 4. *Under the same assumptions of Definition 23, there is an IMLL proof net Θ_1' such that $G(\Theta_1') = \langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'} \rangle$.*

Proof. Let Θ_1' be the mathematical structure obtained from Θ_1 in the following manner:

- deleting $\otimes^+$-link $\frac{\langle A_1^+, 1 \rangle \quad \langle A_2^+, 3 \rangle}{\langle A_1 \otimes A_2{}^+, 2 \rangle}$ and
- adding
  1. ID-link $\frac{}{\langle p^-, -1 \rangle \quad \langle p^+, -2 \rangle}$,
  2. $\otimes^-$-link $\frac{\langle A_2^+, 3 \rangle \quad \langle p^-, -1 \rangle}{\langle A_2 \multimap p^-, 0 \rangle}$,
  3. $\otimes^-$-link $\frac{\langle A_1^+, 1 \rangle \quad \langle A_2 \multimap p^-, 0 \rangle}{\langle A_1 \multimap A_2 \multimap p^-, 2 \rangle}$, and
  4. $\invamp^+$-link $\frac{\langle A_1 \multimap A_2 \multimap p^-, 2 \rangle \quad \langle p^+, -2 \rangle}{\langle (A_1 \multimap A_2 \multimap p) \multimap p^+, -3 \rangle}$.

Then it is obvious that Θ_1' is an IMLL proof structure because Θ_1 is an IMLL proof structure. It is also obvious that $G(\Theta_1') = \langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'} \rangle$.

Next, we prove that Θ_1' is an IMLL proof net. Let S be a Danos–Regnier switching for Θ_1. Then the set of $\invamp^{pl}$-link occurrences in Θ_1' is the same as that of Θ_1. Hence the Danos–Regnier graph Θ_{1S}' is well-defined. Then it is obvious that Θ_{1S} is acyclic and connected iff Θ_{1S}' is acyclic and connected. Since Θ_1 is an IMLL proof net, Θ_1' is also an IMLL proof net. $\square$

Then we write $\Theta_1 \Rightarrow_{\mathrm{PIIT}} \Theta_1'$.

Lemma 5. *Let Θ_1 and Θ_2 be closed IMLL proof nets with the same conclusion $A_1 \otimes A_2{}^+$ such that $\Theta_1 \neq \Theta_2$. If $\Theta_1 \Rightarrow_{\mathrm{PIIT}} \Theta_1'$ and $\Theta_2 \Rightarrow_{\mathrm{PIIT}} \Theta_2'$, then $\Theta_1' \neq \Theta_2'$.*

Proof. We prove the contraposition of the statement, i.e., if $\Theta_1' = \Theta_2'$, then $\Theta_1 = \Theta_2$. When we let $G(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$, $G(\Theta_1') = \langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'}\rangle$, $G(\Theta_2) = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle$, and $G(\Theta_2') = \langle V_2', E_2', \ell_{V_2'}, \ell_{E_2'}\rangle$, the relationship of Definition 23 holds between $\langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$ and $\langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'}\rangle$. Moreover, the same relationship holds $\langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle$ and $\langle V_2', E_2', \ell_{V_2'}, \ell_{E_2'}\rangle$ except for replacing V_1, V_1', E_1, E_1' by V_2, V_2', E_2, E_2' respectively. By the assumption $\Theta_1' = \Theta_2'$, we have a graph isomorphism $\langle h_{V'} : V_1' \to V_2', h_{E'} : E_1' \to E_2'\rangle$. Then we would like to have a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ from $\langle h_{V'}, h_{E'}\rangle$. Let V'_{main} be $\{-3, -2, -1, 0, 1, 2, 3\}$ and E'_{main} be $\{\langle 0, 2\rangle, \langle 2, 1\rangle, \langle -1, 0\rangle, \langle 0, 3\rangle, \langle -2, -1\rangle, \langle -3, 2\rangle, \langle -2, -3\rangle\}$. We prove the following claim.

Claim 1. (1) *$h_{V'}|_{V'_{\mathrm{main}}}$ is the identity map on V'_{main}.*
(2) *$h_{E'}|_{E'_{\mathrm{main}}}$ is the identity map on E'_{main}.*

In order to prove the claim, it is sufficient to prove the following subclaim, since $h_{V'}$ must preserve the positive conclusion between $G(\Theta_1')$ and $G(\Theta_2')$, i.e., $h_{V'}(-3) = -3$.

Subclaim 1. (1) *For any $e \in E_1'$ and $e' \in E_2'$, if $h_{V'}(\mathrm{src}(e)) = \mathrm{src}(e')$, $\ell_{V_1'}(\mathrm{tgt}(e)) = \ell_{V_2'}(\mathrm{tgt}(e'))$, and $\ell_{E_1'}(e) = \ell_{E_2'}(e')$, then $h_{E'}(e) = e'$ and $h_{V'}(\mathrm{tgt}(e)) = \mathrm{tgt}(e')$.*
(2) *For any $e \in E_1'$ and $e' \in E_2'$, if $h_{V'}(\mathrm{tgt}(e)) = \mathrm{tgt}(e')$, $\ell_{V_1'}(\mathrm{src}(e)) = \ell_{V_2'}(\mathrm{src}(e'))$, and $\ell_{E_1'}(e) = \ell_{E_2'}(e')$, then $h_{E'}(e) = e'$ and $h_{V'}(\mathrm{src}(e)) = \mathrm{src}(e')$.*

Proof of Subclaim 1. We only prove (1), since we can prove (2) similarly. First we note $\mathrm{src}(e') = h_{V'}(\mathrm{src}(e)) = \mathrm{src}(h_{E'}(e))$.

- The case where $\ell_{E_1'}(e) = \mathbf{ID}$:
  If $\mathrm{tgt}(e') = h_{V'}(\mathrm{tgt}(e)) = \mathrm{tgt}(h_{E'}(e))$, then
  $e' = \langle \mathrm{src}(e'), \mathrm{tgt}(e')\rangle = \langle \mathrm{src}(h_{E'}(e)), \mathrm{tgt}(h_{E'}(e))\rangle = h_{E'}(e)$.
  Next we assume that $\mathrm{tgt}(e') \neq h_{V'}(\mathrm{tgt}(e)) = \mathrm{tgt}(h_{E'}(e))$. Then since in $G(\Theta_2')$, $\ell_{E_2'}(e') = \ell_{E_1'}(e) = \mathbf{ID}$ and $\ell_{E_2'}(h_{E'}(e)) = \ell_{E_1'}(e) = \mathbf{ID}$, $\mathrm{src}(e')$ is a conclusion of two different ID-links, which correspond to e' and $h_{E'}(e)$. This contradicts Θ_2' being a proof-structure.
- The case where $\ell_{E_1'}(e) = \mathbf{L}$:
  If $\mathrm{tgt}(e') = h_{V'}(\mathrm{tgt}(e)) = \mathrm{tgt}(h_{E'}(e))$, then
  $e' = \langle \mathrm{src}(e'), \mathrm{tgt}(e')\rangle = \langle \mathrm{src}(h_{E'}(e)), \mathrm{tgt}(h_{E'}(e))\rangle = h_{E'}(e)$.
  Next we assume that $\mathrm{tgt}(e') \neq h_{V'}(\mathrm{tgt}(e)) = \mathrm{tgt}(h_{E'}(e))$. Then $\ell_{E_2'}(e') = \ell_{E_1'}(e) = \mathbf{L}$, $\ell_{E_2'}(h_{E'}(e)) = \ell_{E_1'}(e) = \mathbf{L}$, and $\ell_{V_2'}(\mathrm{tgt}(h_{E'}(e))) = \ell_{V_2'}(h_{V'}(\mathrm{tgt}(e))) = \ell_{V_1'}(\mathrm{tgt}(e)) = \ell_{V_2'}(\mathrm{tgt}(e'))$. Hence in Θ_2', the link L corresponding to e' is different from the link L' corresponding to $h_{E'}(e)$. But L must be the same kind as L'. Then $\mathrm{src}(e') = \mathrm{src}(h_{E'}(e))$ is a premise of L and L' or the conclusion of L and L'. This contradicts Θ_2' being a proof-structure.
- The case where $\ell_{E_1'}(e) = \mathbf{R}$:
  Similar to the case above. $\square$

Next we define $h_V : V_1 \to V_2$ and $h_E : E_1 \to E_2$.

$$h_V(v) = \begin{cases} h_{V'}(v) & \text{if } v \in V_1 - \{1,2,3\} (= V_1' - V'_{\mathrm{main}}) \\ v & \text{if } v \in \{1,2,3\} \end{cases}$$

$$h_E(e) = \begin{cases} h_{E'}(e) & \text{if } e \in E_1 - \{\langle 2,1\rangle, \langle 2,3\rangle\} (= E_1' - E'_{\mathrm{main}}) \\ e & \text{if } e \in \{\langle 2,1\rangle, \langle 2,3\rangle\}. \end{cases}$$

The following claim is obvious because $h_{V'}|_{V_1' - V'_{\mathrm{main}}} : (V_1' - V'_{\mathrm{main}}) \to (V_2' - V'_{\mathrm{main}})$ and $h_{E'}|_{E_1' - E'_{\mathrm{main}}} : (E_1' - E'_{\mathrm{main}}) \to (E_2' - E'_{\mathrm{main}})$ are a bijection by Claim 1.

Claim 2. *$h_V : V_1 \to V_2$ and $h_E : E_1 \to E_2$ are a bijection.*

Claim 3. *For any $v \in V_1$, $\ell_{V_1}(v) = \ell_{V_2}(h_V(v))$.*

Proof of Claim 3. • The case where $v \in V_1 - \{1, 2, 3\}$:
We note $V_1' - V_{\mathrm{main}}' = V_1 - \{1, 2, 3\}$ and $V_2' - V_{\mathrm{main}}' = V_2 - \{1, 2, 3\}$. Then $h_V(v) \in V_2 - \{1, 2, 3\}$ since $h_V : (V_1 - \{1, 2, 3\}) \uplus \{1, 2, 3\} \to (V_2 - \{1, 2, 3\}) \uplus \{1, 2, 3\}$ is a bijection. Then $\ell_{V_1}(v) = \ell_{V_1'}(v) = \ell_{V_2'}(h_{V'}(v)) = \ell_{V_2'}(h_V(v)) = \ell_{V_2}(h_V(v))$ since for any $v \in V_1'$, $\ell_{V_1'}(v) = \ell_{V_2'}(h_{V'}(v))$.

• The case where $v \in \{1, 2, 3\}$:
If $v = 1$, then $\ell_{V_1}(1) = A_1^+ = \ell_{V_2}(1) = \ell_{V_2}(h_V(1))$. The rest are similar. □

Claim 4. *For any $e \in E_1$, the following holds:*

(1) $h_V(\mathrm{src}(e)) = \mathrm{src}(h_E(e))$;
(2) $h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_E(e))$;
(3) $\ell_{E_1}(e) = \ell_{E_2}(h_E(e))$.

Proof of Claim 4. • The case where $e \in E_1 - \{\langle 2, 1\rangle, \langle 2, 3\rangle\}$:

(1) We note $E_1' - E_{\mathrm{main}}' = E_1 - \{\langle 2, 1\rangle, \langle 2, 3\rangle\}$ and $E_2' - E_{\mathrm{main}}' = E_2 - \{\langle 2, 1\rangle, \langle 2, 3\rangle\}$. Then the following two cases are considered:
   • The case where $\mathrm{src}(e) \in \{1, 2, 3\}$:
   We let $\mathrm{src}(e)$ be 1. Then $h_V(\mathrm{src}(e)) = h_V(1) = 1 = h_{V'}(1) = h_{V'}(\mathrm{src}(e)) = \mathrm{src}(h_{E'}(e)) = \mathrm{src}(h_E(e))$ since for any $e \in E_1'$, $h_{V'}(\mathrm{src}(e)) = \mathrm{src}(h_{E'}(e))$. The case where $\mathrm{src}(e)$ is 2 or 3 is similar.
   • The case where $\mathrm{src}(e) \notin \{1, 2, 3\}$:
   $h_V(\mathrm{src}(e)) = h_{V'}(\mathrm{src}(e)) = \mathrm{src}(h_{E'}(e)) = \mathrm{src}(h_E(e))$ since for any $e \in E_1'$, $h_{V'}(\mathrm{src}(e)) = \mathrm{src}(h_{E'}(e))$.

(2) Similar to the case above.

(3) We note $h_E(e) \in E_2 - \{\langle 2, 1\rangle, \langle 2, 3\rangle\}$ since

$$\begin{aligned} h_E \ &: (E_1 - \{\langle 2, 1\rangle, \langle 2, 3\rangle\}) \uplus \{\langle 2, 1\rangle, \langle 2, 3\rangle\} \\ &\to (E_2 - \{\langle 2, 1\rangle, \langle 2, 3\rangle\}) \uplus \{\langle 2, 1\rangle, \langle 2, 3\rangle\} \end{aligned}$$

is a bijection. Then $\ell_{E_1}(e) = \ell_{E_1'}(e) = \ell_{E_2'}(h_{E'}(e)) = \ell_{E_2'}(h_E(e)) = \ell_{E_2}(h_E(e))$, since for any $e \in E_1'$, $\ell_{E_1'}(e) = \ell_{E_2'}(h_{E'}(e))$.

• The case where $e \in \{\langle 2, 1\rangle, \langle 2, 3\rangle\}$:
Let e be $\langle 2, 1\rangle$. Then

(1) $h_V(\mathrm{src}(\langle 2, 1\rangle)) = h_V(2) = 2 = \mathrm{src}(\langle 2, 1\rangle) = \mathrm{src}(h_E(\langle 2, 1\rangle))$;
(2) $h_V(\mathrm{tgt}(\langle 2, 1\rangle)) = h_V(1) = 1 = \mathrm{tgt}(\langle 2, 1\rangle) = \mathrm{tgt}(h_E(\langle 2, 1\rangle))$;
(3) $\ell_{E_1}(\langle 2, 1\rangle) = \mathbf{L} = \ell_{E_2}(\langle 2, 1\rangle) = \ell_{E_2}(h_E(\langle 2, 1\rangle))$.

The case where $e = \langle 2, 3\rangle$ is similar. □

From Claims 2–4, $\langle h_V, h_E\rangle$ is a graph isomorphism. □

Lemma 6. *Let Θ_1 be a normal IMLL proof net with the positive conclusion $A_1 \otimes A_2{}^+$. Moreover we assume that $\Theta_1 \Rightarrow_{\mathrm{PIIT}} \Theta_1'$. Then there is a wrapping net $C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[]$ such that $C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[\Theta_1] = \Theta_1'$.*

Proof. We just let $C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[]$ be the wrapping net shown in Fig. 18. □

Proposition 11. *Let Θ_1 and Θ_2 be closed IMLL proofs with the same conclusion A^+ such that $\Theta_1 \neq \Theta_2$. Then there is a wrapping net $C^{A^+}_{A_0^+}[]$ such that A_0^+ is a pseudo IIMLL formula and $C^{A^+}_{A_0^+}[\Theta_1] \neq C^{A^+}_{A_0^+}[\Theta_2]$.*

Proof. (1) The case where A^+ is a pseudo IIMLL formula:
We choose $C^{A^+}_{A_+}[]$ to be the η-expansion of the ID-link with the conclusions A^+ and A^-. Then $C^{A^+}_{A_+}[\Theta_1] = \Theta_1 \neq \Theta_2 = C^{A^+}_{A_+}[\Theta_1]$.

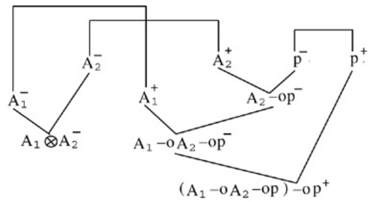


Fig. 18. A wrapping net.

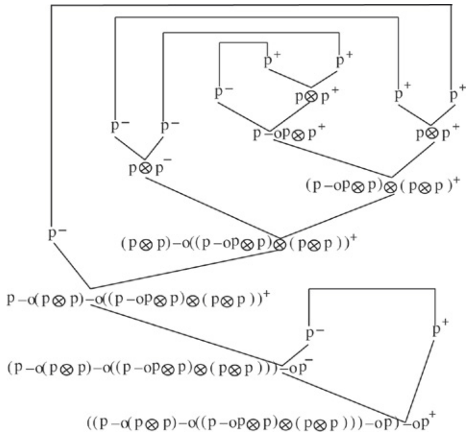


Fig. 19. Θ_1^b.

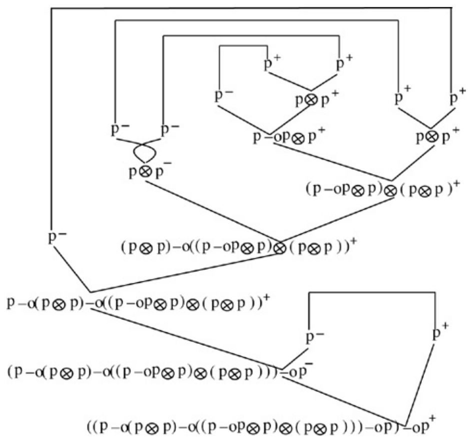


Fig. 20. Θ_2^b.

(2) Otherwise:
Then $A^+ = A_1 \otimes A_2{}^+$ for some formulas A_1^+ and A_2^+. We let the wrapping net we are looking for be $C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[]$ obtained from Lemma 6. Then since $\Theta_1 \Rightarrow_{\mathrm{PIIT}} C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[\Theta_1]$ and $\Theta_2 \Rightarrow_{\mathrm{PIIT}} C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[\Theta_2]$, we have $C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[\Theta_1] \neq C^{A_1 \otimes A_2{}^+}_{(A_1 \multimap A_2 \multimap p) \multimap p^+}[\Theta_1]$ from Lemma 5. $\square$

Example 3. Applying $\Rightarrow_{\mathrm{PIIT}}$ to two IMLL proof nets Θ_1^a and Θ_2^a of Figs. 14 and 15, we obtain two IMLL proof nets Θ_1^b and Θ_2^b of Figs. 19 and 20 respectively. We note that $\Theta_1^b \neq \Theta_2^b$ and the conclusion of Θ_1^b and Θ_2^b is a pseudo IIMLL formula. Following Proposition 11 we can obtain a wrapping net $C[]$ such that $C[\Theta_1^a] = \Theta_1^b \neq \Theta_2^b = C[\Theta_2^a]$ (but we omit this).

3.2. Step 2: The linear distributive reductions

In order to define the linear distributive reductions we need to define linear replacements on IMLL formulas.

Definition 24 (*Linear Replacements*). Let A^{pl} be an IMLL formula and B be a subformula occurrence of A. (So we suppose that B has locative information in A^{pl}.) Then when we assume C is a negative-free MLL formula, we define $A\langle C/B\rangle^{pl}$ to the IMLL formula obtained from A replacing the occurrence B by C.

Note that $A\langle C/B\rangle^{pl}$ is different from the usual substitution $A[C/B]^{pl}$, in which several occurrences B in A^{pl} are replaced.

Definition 25 (*The Polarity of a Subformula Occurrence of an IMLL Formula*). Let A^{pl} be an IMLL formula and A_0 be a subformula occurrence of A^{pl}. Then we define the polarity of A_0 in A^{pl} (denoted by $\mathrm{pol}(A_0, A^{pl}) \in \{+, -\}$) inductively:

(1) The case where $A^{pl} = p^{pl}$: $\mathrm{pol}(p, A^{pl}) = pl$;
(2) The case where $A^{pl} = B \multimap C^{pl}$:
  (a) if $A_0 = B \multimap C$ then $\mathrm{pol}(A_0, A^{pl}) = pl$;
  (b) If A_0 occurs in B, then $\mathrm{pol}(A_0, A^{pl}) = \mathrm{pol}(A_0, B^{\overline{pl}})$, where $\overline{pl} = +$ if $pl = -$ and $\overline{pl} = -$ otherwise;
  (c) If A_0 occurs in C, then $\mathrm{pol}(A_0, A^{pl}) = \mathrm{pol}(A_0, C^{pl})$;
(3) The case where $A^{pl} = B \otimes C^{pl}$:
  (a) if $A_0 = B \otimes C$ then $\mathrm{pol}(A_0, A^{pl}) = pl$;
  (b) If A_0 occurs in B, then $\mathrm{pol}(A_0, A^{pl}) = \mathrm{pol}(A_0, B^{pl})$;
  (c) If A_0 occurs in C, then $\mathrm{pol}(A_0, A^{pl}) = \mathrm{pol}(A_0, C^{pl})$.

The following proposition ensures the consistency of linear replacements about the polarities.

Proposition 12. *Let A^{pl} be an IMLL formula, B be a subformula occurrence of A, and C be a negative-free MLL formula. Then,* $\mathrm{pol}(C, A\langle C/B\rangle^{pl}) = \mathrm{pol}(B, A^{pl})$.

Proof. We prove this proposition by induction on the structure of A.

(1) The case where $A = p$:
  In the case B must be p.
  (a) The case where $C = p$:
    Then, $\mathrm{pol}(C, A\langle C/B\rangle^{pl}) = \mathrm{pol}(p, p\langle p/p\rangle^{pl}) = \mathrm{pol}(p, p^{pl}) = \mathrm{pol}(B, A^{pl})$.
  (b) The case where $C = C_1 \multimap C_2$:
    $\mathrm{pol}(C, A\langle C/B\rangle^{pl}) = \mathrm{pol}(C_1 \multimap C_2, p\langle C_1 \multimap C_2/p\rangle^{pl}) = \mathrm{pol}(C_1 \multimap C_2, C_1 \multimap C_2{}^{pl}) = pl = \mathrm{pol}(p, p^{pl}) = \mathrm{pol}(B, A^{pl})$.
  (c) The case where $C = C_1 \otimes C_2$:
    $\mathrm{pol}(C, A\langle C/B\rangle^{pl}) = \mathrm{pol}(C_1 \otimes C_2, p\langle C_1 \otimes C_2/p\rangle^{pl}) = \mathrm{pol}(C_1 \otimes C_2, C_1 \otimes C_2{}^{pl}) = pl = \mathrm{pol}(p, p^{pl}) = \mathrm{pol}(B, A^{pl})$.
(2) The case where $A = A_1 \multimap A_2$:
  (a) The case where B occurs in A_1:
    $\mathrm{pol}(C, A_1 \multimap A_2\langle C, B\rangle^{pl}) = \mathrm{pol}(C, A_1\langle C/B\rangle \multimap A_2{}^{pl}) = \mathrm{pol}(C, A_1\langle C/B\rangle^{\overline{pl}}) = \mathrm{pol}(B, A_1^{\overline{pl}})$ (inductive hypothesis) $= \mathrm{pol}(B, A_1 \multimap A_2{}^{pl})$.
  (b) The case where B occurs in A_2:
    $\mathrm{pol}(C, A_1 \multimap A_2\langle C, B\rangle^{pl}) = \mathrm{pol}(C, A_1 \multimap (A_2\langle C/B\rangle)^{pl}) = \mathrm{pol}(C, A_2\langle C/B\rangle^{pl}) = \mathrm{pol}(B, A_2^{pl})$ (inductive hypothesis) $= \mathrm{pol}(B, A_1 \multimap A_2{}^{pl})$.
(3) The case where $A = A_1 \otimes A_2$:
  (a) The case where B occurs in A_1:
    $\mathrm{pol}(C, A_1 \otimes A_2\langle C, B\rangle^{pl}) = \mathrm{pol}(C, (A_1\langle C/B\rangle) \multimap A_2{}^{pl}) = \mathrm{pol}(C, A_1\langle C/B\rangle^{pl}) = \mathrm{pol}(B, A_1^{pl})$ (inductive hypothesis) $= \mathrm{pol}(B, A_1 \otimes A_2{}^{pl})$.
  (b) The case where B occurs in A_2:
    Similar to the case (3a) above. □

Moreover, we need the notion of bound paths on normal IMLL proof nets.

Definition 26 (*General Paths*). Let $G = \langle V, E\rangle$ be a directed graph. Then we define E^r be the set E with two functions $\mathrm{src}^r : E^r \to V$ and $\mathrm{tgt}^r : E^r \to V$ such that $\mathrm{src}^r(e) = \mathrm{tgt}(e)$ and $\mathrm{tgt}^r(e) = \mathrm{src}(e)$. In the following when $e \in E$, we write the element of E^r corresponding to e by e^r, $\mathrm{src}^r(e)$ by $\mathrm{src}(e^r)$, and $\mathrm{tgt}^r(e)$ by $\mathrm{tgt}(e^r)$. Then a general path $f_1, \ldots, f_n$ $(n \geq 0)$ of $G = \langle V, E\rangle$ is a sequence of $E \cup E^r$ such that $\mathrm{src}(f_{i+1}) = \mathrm{tgt}(f_i)$ for each i $(1 \leq i \leq n-1)$.

Definition 27 (*Bound Paths on Normal IMLL Proof Nets*). Let Θ be a normal IMLL proof net and $G(\Theta)$ be $\langle V, E, \ell_V, \ell_E\rangle$. Moreover we assume that $\langle F_0, b_0\rangle$ is a conclusion of Θ and $\langle F, b\rangle$ is a formula occurrence of Θ. Then a general path $f_1, \ldots, f_n$ $(n \geq 0)$ is a bound path starting at b and ending at b_0 if the following conditions are satisfied:

(1) $\mathrm{src}(f_1) = b$;
(2) $\mathrm{tgt}(f_n) = b_0$;
(3) for each i $(1 \leq i \leq n)$, there is a link L_i in Θ such that $\langle \ell_V(\mathrm{tgt}(f_i)), \mathrm{tgt}(f_i)\rangle$ is the conclusion of L_i and $\langle \ell_V(\mathrm{src}(f_i)), \mathrm{src}(f_i)\rangle$ is a premise of L_i.

Proposition 13 (*Uniqueness of a Bound Path*). *We put the same assumptions as that of Definition 27. Then if $f_1, \ldots, f_n$ and $f'_1, \ldots, f'_{n'}$ are two bound paths starting at b and ending at b_0 in Θ, then $n = n'$ and for each i $(1 \leq i \leq n)$, $f_i = f'_i$.*

Proof. We prove by induction on lengths n of bound paths $f_1, \ldots, f_n$.

(1) The case where $n = 1$:
Then $\langle F, \mathrm{src}(f_1)\rangle = \langle F, \mathrm{src}(f'_1)\rangle = \langle F, b\rangle$ and $\langle F_0, \mathrm{tgt}(f_1)\rangle = \langle F_0, \mathrm{tgt}(f'_1)\rangle = \langle F_0, b_0\rangle$, which is the conclusion of Θ. Then by Proposition 4 $f_1 = f'_1$.
(2) The case where $n > 1$:
Then $\langle F, \mathrm{src}(f_1)\rangle = \langle F, \mathrm{src}(f'_1)\rangle = \langle F, b\rangle$. If $\mathrm{tgt}(f_1) \neq \mathrm{tgt}(f'_1)$, then $\langle F, b\rangle$ is a premise of several links. But this contradicts Θ being a proof structure. Hence $\mathrm{tgt}(f_1) = \mathrm{tgt}(f'_1)$. By Proposition 4 $f_1 = f'_1$. Then applying inductive hypothesis to $f_2, \ldots, f_n$ and $f'_2, \ldots, f'_{n'}$, we find that $n = n'$ and for each i $(2 \leq i \leq n)$, $f_i = f'_i$. □

We note that the proposition above does not hold without the assumption about bound paths. That is, in general, in a normal IMLL proof net there may be several general paths with the same starting point and ending point.

Definition 28. Let Θ be a normal IMLL proof net. Let L_1 and L_2 be two link occurrences in Θ. We assume that the conclusion of L_1 is $\langle F_1, b_1\rangle$ and that of L_2 is $\langle F_2, b_2\rangle$. Then L_1 is above L_2 if there is a conclusion $\langle F_0, b_0\rangle$ in $G(\Theta)$ such that the bounded path starting with b_1 and ending with b_0 passes b_2.

Definition 29 (*Linear Distributive Reductions*). Let Θ_1 be a normal IMLL proof net with the positive conclusion A^+ such that any of the following conditions holds:

(1) Θ_1 has a subformula occurrence with the form $(C_1 \multimap C_2) \multimap B$ such that $\mathrm{pol}((C_1 \multimap C_2) \multimap B, A^+) = -$.
(2) Θ_1 has a subformula occurrence with the form $C \multimap B_1 \otimes B_2$ such that $\mathrm{pol}(C \multimap B_1 \otimes B_2, A^+) = -$.
(3) Θ_1 has a subformula occurrence with the form $(C \multimap B_1) \otimes B_2$ such that $\mathrm{pol}((C \multimap B_1) \otimes B_2, A^+) = -$.
(4) Θ_1 has a subformula occurrence with the form $B_1 \otimes (C \multimap B_2)$ such that $\mathrm{pol}(B_1 \otimes (C \multimap B_2), A^+) = -$.

Let $G(\Theta_1)$ be $\langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$. Then $G(\Theta_1)$ must include the left side of any of Figs. 21–25. From Proposition 13 such a subgraph is uniquely determined. According to the inclusion, let $\langle V'_1, E'_1, \ell_{V'_1}, \ell_{E'_1}\rangle$ be the labeled directed graph obtained from $G(\Theta_1)$ replacing the left side of Figs. 21–24, or 25 by the right side of the same figure.

Lemma 7. *Under the same assumptions of Definition 29, there is an IMLL proof net Θ'_1 such that $G(\Theta'_1) = \langle V'_1, E'_1, \ell_{V'_1}, \ell_{E'_1}\rangle$.*

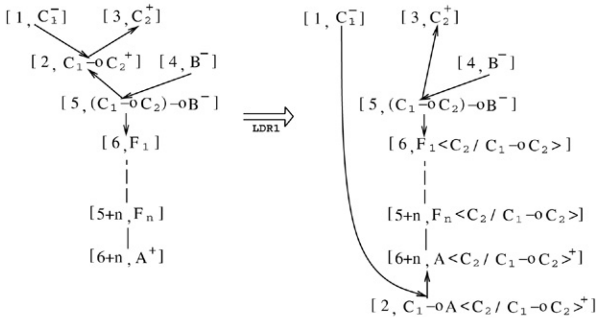


Fig. 21. LDR_1 transformation.

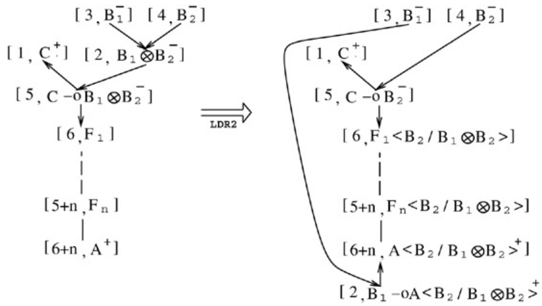


Fig. 22. LDR_2 transformation.

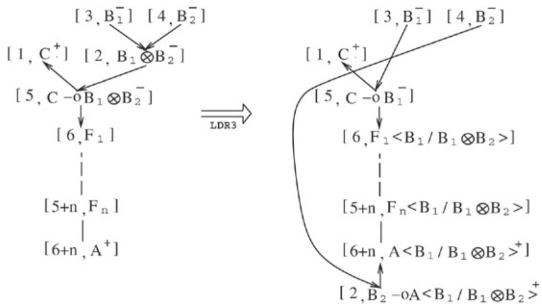


Fig. 23. LDR_3 transformation.

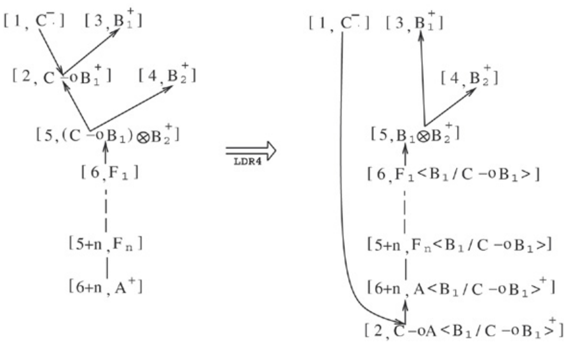


Fig. 24. LDR_4 transformation.

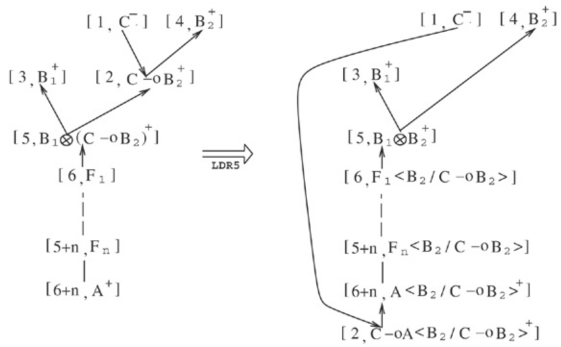


Fig. 25. LDR_5 transformation.

Proof. We only consider the case where the transformation of Fig. 21 is applied. The other cases are similar. Let Θ_1' be the mathematical structure obtained from Θ_1 in the following manner:

- deleting
  (1) $\wp^+$-link L_1 : $\frac{\langle C_1^-,1\rangle \quad \langle C_2^+,3\rangle}{\langle C_1 \multimap C_2{}^+,2\rangle}$ and
  (2) $\otimes^-$-link L_2 : $\frac{\langle C_1 \multimap C_2{}^+,2\rangle \quad \langle B^-,4\rangle}{\langle (C_1 \multimap C_2) \multimap B^-,5\rangle}$,
- adding
  (1) $\wp^+$-link L_3 : $\frac{\langle C_1^-,1\rangle \quad \langle A\langle C_2/C_1 \multimap C_2\rangle^+,6+n\rangle}{\langle C_1 \multimap A\langle C_2/C_1 \multimap C_2\rangle^+,2\rangle}$ and
  (2) $\otimes^-$-link L_4 : $\frac{\langle C_2^+,3\rangle \quad \langle B^-,4\rangle}{\langle (C_1 \multimap C_2) \multimap B^-,5\rangle}$, and
- replacing F_i by $F_i\langle C_2/C_1 \multimap C_2\rangle$ for each i $(1 \le i \le n)$.

Then it is obvious that Θ_1' is an IMLL proof structure because Θ_1 is an IMLL proof structure. It is also obvious that $G(\Theta_1') = \langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'}\rangle$.

Next, we prove that Θ_1' is an IMLL proof net. Let S' be a Danos–Regnier switching for Θ_1'. Then we define a Danos–Regnier switching S for Θ_1 in the following way:

$$S(L) = \begin{cases} S'(L) & \text{if } L \neq L_1 \\ \mathbf{R} & \text{if } L = L_1. \end{cases}$$

Let G_S be the graph obtained from the Danos–Regnier graph Θ_{1S} identifying $\langle C_1 \multimap C_2{}^+, 2\rangle$ and $\langle C_2^+, 3\rangle$. Then,

(1) The case where $S'(L_3) = \mathbf{L}$:
Let $G'_{S'}$ be the graph obtained from $\Theta'_{1S'}$ identifying $\langle C_1^-, 1\rangle$ and $\langle C_2 \multimap A\langle C_2/C_1 \multimap C_2\rangle^+, 2\rangle$, and
(2) The case where $S'(L_3) = \mathbf{R}$:
Let $G'_{S'}$ be the graph obtained from $\Theta'_{1S'}$ identifying $\langle A\langle C_2/C_1 \multimap C_2\rangle^+, 6+n\rangle$ and $\langle C_2 \multimap A\langle C_2/C_1 \multimap C_2\rangle^+, 2\rangle$.

In both cases, it is obvious that there is a graph isomorphism between G_S and $G'_{S'}$ (forgetting the information about labels, i.e., IMLL formulas). If $\Theta_{1S'}$ has a cycle or disconnected components, then $G'_{S'}$ also has a cycle or disconnected components and hence G_S also has. Then Θ_{1S} also has a cycle or disconnected components. This contradicts Θ_1 being a proof net. □

Then, we write

(1) $\Theta_1 \Rightarrow_{\mathrm{LDR}_1} \Theta_1'$ if the transformation of Fig. 21 is applied;
(2) $\Theta_1 \Rightarrow_{\mathrm{LDR}_2} \Theta_1'$ if the transformation of Fig. 22 is applied;
(3) $\Theta_1 \Rightarrow_{\mathrm{LDR}_3} \Theta_1'$ if the transformation of Fig. 23 is applied;
(4) $\Theta_1 \Rightarrow_{\mathrm{LDR}_4} \Theta_1'$ if the transformation of Fig. 24 is applied, and
(5) $\Theta_1 \Rightarrow_{\mathrm{LDR}_5} \Theta_1'$ if the transformation of Fig. 25 is applied.

Note that $\bigcup_{1\le i\le 5} \Rightarrow_{\mathrm{LDR}_i}$ is not confluent because there are IMLL proof nets Θ_1 and Θ_2 such that $\Theta \Rightarrow_{\mathrm{LDR}_2} \Theta_1$, $\Theta \Rightarrow_{\mathrm{LDR}_3} \Theta_2$, and Θ_1 and Θ_2 have different normal forms w.r.t. the reduction relation $\bigcup_{1\le i\le 5} \Rightarrow_{\mathrm{LDR}_i}$.

Lemma 8. *Let Θ_1 and Θ_2 be closed IMLL proof nets with the same positive conclusion such that $\Theta_1 \neq \Theta_2$. If there are IMLL proof nets Θ_1' and Θ_2' with the same positive conclusion such that $\Theta_1 \Rightarrow_{\mathrm{LDR}_i} \Theta_1'$ and $\Theta_2 \Rightarrow_{\mathrm{LDR}_i} \Theta_2'$, then $\Theta_1' \neq \Theta_2'$ for each i $(1 \le i \le 5)$.*

Proof. Basically the same line as the proof of Lemma 5. There is nothing interesting here. □

Lemma 9. *Let Θ_1 be a normal IMLL proof net with the positive conclusion A^+. We assume $\Theta_1 \Rightarrow_{\mathrm{LDR}_i} \Theta_1'$ for some i $(1 \le i \le 5)$. Let A'^+ be the positive conclusion of Θ_1'. Then there is a $C^{A^+}_{A'^+}[]$ such that $C^{A^+}_{A'^+}[\Theta_1] = \Theta_1'$.*

Proof. We only consider the case where $\Theta_1 \Rightarrow_{\mathrm{LDR}_1} \Theta_1'$. The other cases are similar. In this case there is a subformula occurrence with the form $(C_1 \multimap C_2) \multimap B$ in Θ_1 such that $\mathrm{pol}((C_1 \multimap C_2) \multimap B, A^+) = -$. Let Π_1 be the η-expansion of the IMLL proof net consisting of exactly one ID-link with the conclusions A^- and A^+. Since Π_1 has the positive conclusion A^+, the linear distributive transformation of Fig. 21 can be applied to Π_1. Let $C^{A^+}_{A'^+}[]$ be the IMLL proof net such that $\Pi_1 \Rightarrow_{\mathrm{LDR}_1} C^{A^+}_{A'^+}[]$. Then it is obvious that $C^{A^+}_{A'^+}[\Theta_1] = \Theta_1'$. □

Proposition 14. *Let Θ_1 and Θ_2 be closed pseudo IIMLL proof nets with the same positive conclusion A^+ such that $\Theta_1 \neq \Theta_2$ and A^+ is not an essentially third-order formula. There is a wrapping net $C[]$ such that $\Theta_1 \Rightarrow_{\mathrm{LDR}_i} C[\Theta_1]$, $\Theta_2 \Rightarrow_{\mathrm{LDR}_i} C[\Theta_2]$, and $C[\Theta_1] \neq C[\Theta_2]$ for some i $(1 \le i \le 5)$.*

Proof. If A^+ is not an essentially third-order formula, then it is obvious that any of linear distributive transformations can be applied. Hence there are IMLL proof nets Θ_1' and Θ_2' such that $\Theta_1 \Rightarrow_i \Theta_1'$ and $\Theta_2 \Rightarrow_i \Theta_2'$ for some i $(1 \le i \le 5)$. From Lemma 8, we can see that $\Theta_1' \neq \Theta_2'$. Moreover from Lemma 9 we can find wrapping nets $C_1[]$ and $C_2[]$ such that $C_1[\Theta_1] = \Theta_1'$ and $C_2[\Theta_2] = \Theta_2'$. But since from the proof of Lemma 9 we can see that the way of constructing wrapping nets $C_1[]$ and $C_2[]$ only depends on the positive conclusion of A^+, we can choose a wrapping net $C[]$ such that $C[] = C_1[] = C_2[]$. Then $C[\Theta_1] \neq C[\Theta_2]$. □

Corollary 1. *Let Θ_1 and Θ_2 be closed pseudo IIMLL proof nets with the same positive conclusion A^+ such that $\Theta_1 \neq \Theta_2$. Then there is a wrapping net $C^{A^+}_{A_0^+}[]$ such that A_0^+ is an essentially third-order formula and $C^{A^+}_{A_0^+}[\Theta_1] \neq C^{A^+}_{A_0^+}[\Theta_2]$.*

Proof. If A^+ is an essentially third-order formula, then we let A_0^+ be A^+ and $C^{A^+}_{A_0^+}[]$ be the η-expansion of the IMLL proof net exactly consisting of one ID-link with the conclusions A^- and A^+.

Next we consider the case where A^+ is not an essentially third-order formula. Then for each normal IMLL proof net Θ, we define $m_{\mathrm{LDR}}(\Theta)$ to be

$$\sum_{L \text{ is a } \otimes^+\text{-link or } \otimes^-\text{-link}} |\{L' \mid L' \text{ is above } L \wedge L' \text{ is a } \parr^+\text{-link or } \parr^-\text{-link}\}|.$$

Moreover we define $m_{\mathrm{LDR}}(\langle\Theta_1, \Theta_2\rangle)$ to be $m_{\mathrm{LDR}}(\Theta_1) + m_{\mathrm{LDR}}(\Theta_2)$. If we obtain $\langle C[\Theta_1], C[\Theta_2]\rangle$ from $\langle\Theta_1, \Theta_2\rangle$ applying Proposition 14, it is obvious that $m_{\mathrm{LDR}}(\langle C[\Theta_1], C[\Theta_2]\rangle) < m_{\mathrm{LDR}}(\langle\Theta_1, \Theta_2\rangle)$. By repeating the procedure, we can obtain a list of wrapping terms $C_1[], \ldots, C_n[]$ such that $C_n[\cdots[C_1[\Theta_1]]\cdots] \neq C_n[\cdots[C_1[\Theta_2]]\cdots]$ and $m_{\mathrm{LDR}}(\langle C_n[\cdots[C_1[\Theta_1]]\cdots], C_n[\cdots[C_1[\Theta_2]]\cdots]\rangle) = 0$. It is obvious that the positive conclusion of $C_n[\cdots[C_1[\Theta_1]]\cdots]$ and $C_n[\cdots[C_1[\Theta_2]]\cdots]$ is an essentially third-order formula. Moreover applying Lemma 3 repeatedly, we can obtain a wrapping net $C[]$ such that $C[] = C_n[\cdots[C_1[]]\cdots]$. So, $C[\Theta_1] \neq C[\Theta_2]$. □

Example 4. Figs. 26 and 27 show two proof nets Θ_1^c and Θ_1^c such that $\Theta_1^b \Rightarrow_{\mathrm{LDR}_4}\Rightarrow_{\mathrm{LDR}_1}\Rightarrow_{\mathrm{LDR}_1} \Theta_1^c$ and $\Theta_2^b \Rightarrow_{\mathrm{LDR}_4}\Rightarrow_{\mathrm{LDR}_1}\Rightarrow_{\mathrm{LDR}_1} \Theta_2^c$, where Θ_1^b and Θ_2^b are proof nets shown in Figs. 19 and 20 respectively. We note that $\Theta_1^c \neq \Theta_2^c$ and the conclusion of Θ_1^c and Θ_2^c is an essentially third-order formula. Following Corollary 1 we obtain a wrapping net $C[]$ such that $C[\Theta_1^b] = \Theta_1^c \neq \Theta_2^c = C[\Theta_2^b]$ (but we omit this).

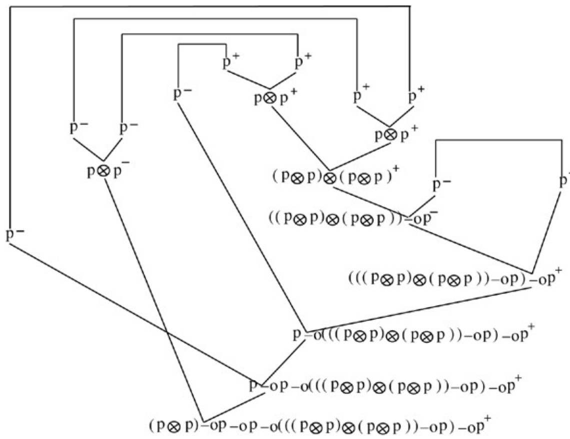


Fig. 26. Θ_1^c.

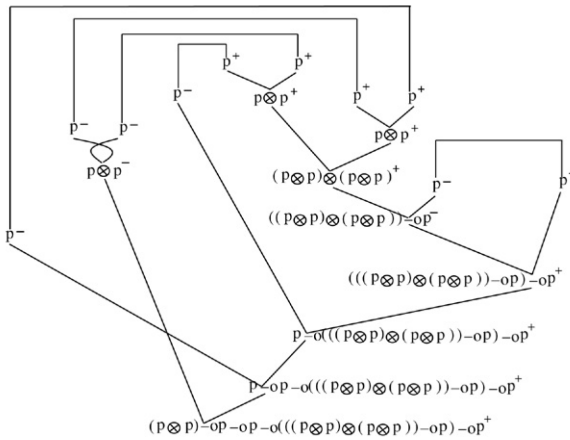


Fig. 27. Θ_2^c.

3.3. Step 3: The Curry transformations

First we describe general statements about the Curry transformations.

Definition 30 (*The Curry Transformations*). Let Θ_1 be a normal IMLL proof net with the positive conclusion A^+ such that any of the following conditions holds:

(1) Θ_1 has a subformula occurrence with the form $(C_1 \otimes C_2) \multimap B$ such that $\mathrm{pol}((C_1 \otimes C_2) \multimap B, A^+) = -$.
(2) Θ_1 has a subformula occurrence with the form $(C_1 \otimes C_2) \multimap B$ such that $\mathrm{pol}((C_1 \otimes C_2) \multimap B, A^+) = +$.

Let $G(\Theta_1)$ be $\langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$. Then $G(\Theta_1)$ must include the left side of any of Figs. 28–31. From Proposition 13 such a subgraph is uniquely determined. According to the inclusion, let $\langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'}\rangle$ be the labeled directed graph obtained from $G(\Theta_1)$ replacing the left side of Figs. 28–30, or 31 by the right side of the same figure.

Lemma 10. *Under the same assumptions of Definition 30, there is an IMLL proof net Θ_1' such that $G(\Theta_1') = \langle V_1', E_1', \ell_{V_1'}, \ell_{E_1'}\rangle$.*

Proof. Similar to the proof of Lemma 4. There is nothing interesting here. □

Then, we write

(1) $\Theta_1 \Rightarrow_{\mathrm{Curry}_1} \Theta_1'$ if the transformation of Fig. 28 is applied;
(2) $\Theta_1 \Rightarrow_{\mathrm{Curry}_2} \Theta_1'$ if the transformation of Fig. 29 is applied;

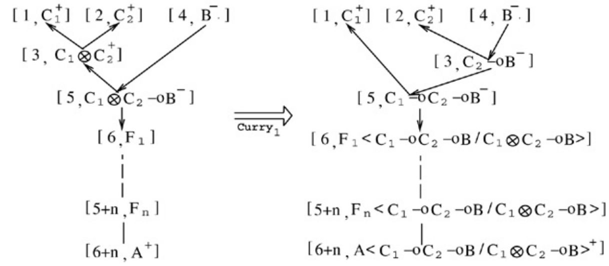


Fig. 28. Curry_1 transformation.

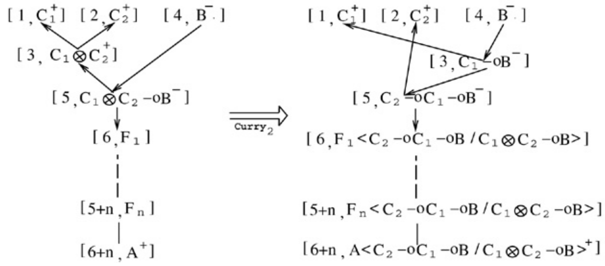


Fig. 29. Curry_2 transformation.

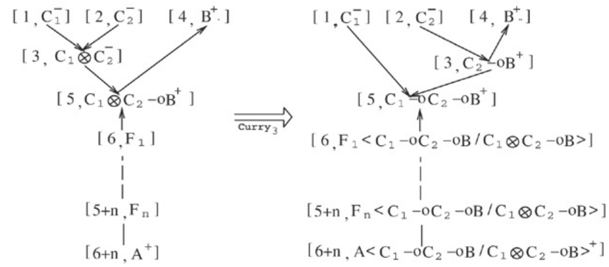


Fig. 30. Curry_3 transformation.

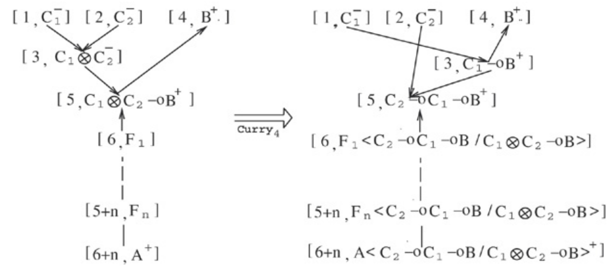


Fig. 31. Curry_4 transformation.

(3) $\Theta_1 \Rightarrow_{\text{Curry}_3} \Theta'_1$ if the transformation of Fig. 30 is applied, and
(4) $\Theta_1 \Rightarrow_{\text{Curry}_4} \Theta'_1$ if the transformation of Fig. 31 is applied.

Lemma 11. *Let Θ_1 and Θ_2 be closed IMLL proof nets with the same positive conclusion such that $\Theta_1 \neq \Theta_2$. If there are IMLL proof nets Θ'_1 and Θ'_2 the same positive conclusion such that $\Theta_1 \Rightarrow_{\text{Curry}_i} \Theta'_1$ and $\Theta_2 \Rightarrow_{\text{Curry}_i} \Theta'_2$, then $\Theta'_1 \neq \Theta'_2$ for each i $(1 \leq i \leq 4)$.*

Proof. Basically the same line as the proof of Lemma 5. There is nothing interesting here. □

Lemma 12. *Let Θ_1 be a normal IMLL proof net with the positive conclusion A^+. We assume $\Theta_1 \Rightarrow_{\text{Curry}_i} \Theta'_1$ for some i $(1 \leq i \leq 4)$. Let A'^+ be the positive conclusion of Θ'_1. Then there is a $C^{A^+}_{A'^+}[]$ such that $C^{A^+}_{A'^+}[\Theta_1] = \Theta'_1$.*

Proof. (1) The case where $\Theta_1 \Rightarrow_{\text{Curry}_i} \Theta_1'$ and $i = 1$ or $i = 2$:

Then there is a subformula occurrence with the form $(C_1 \otimes C_2) \multimap B$ in Θ_1 such that $\text{pol}((C_1 \otimes C_2) \multimap B, A^+) = -$. Let Π_1 be the η-expansion of the IMLL proof net consisting of exactly one ID-link with the conclusions A^- and A^+. Since Π_1 has the positive conclusion A^+, a Curry transformation of Fig. 28 or 29 can be applied to Π_1. Let $C^{A^+}_{A'^+}[]$ be the IMLL proof net such that $\Pi_1 \Rightarrow_{\text{Curry}_i} C^{A^+}_{A'^+}[]$, where $i = 1$ or $i = 2$. Then it is obvious that $C^{A^+}_{A'^+}[\Theta_1] = \Theta_1'$.

(2) The case where $\Theta_1 \Rightarrow_{\text{Curry}_i} \Theta_1'$ and $i = 2$ or $i = 3$:

Similar to the case above. □

Proposition 15. *Let Θ_1 and Θ_2 be closed pseudo IIMLL proof nets with the same positive conclusion A^+ such that $\Theta_1 \neq \Theta_2$, A^+ has a subformula occurrence with the form $B_1 \otimes B_2 \multimap p$, and* $\text{pol}(B_1 \otimes B_2 \multimap p, A^+) = -$ *or* $\text{pol}(B_1 \otimes B_2 \multimap p, A^+) = +$, *where B_1 and B_2 are constructed from several occurrences of p using only $\otimes$-connectives. There is a wrapping net $C[]$ such that $\Theta_1 \Rightarrow_{\text{Curry}_i} C[\Theta_1]$, $\Theta_2 \Rightarrow_{\text{Curry}_i} C[\Theta_2]$, and $C[\Theta_1] \neq C[\Theta_2]$ for some i $(1 \le i \le 4)$.*

Proof. (1) The case where $\text{pol}(B_1 \otimes B_2 \multimap p, A^+) = -$:

In this case, it is obvious that Curry_1 transformation can be applied. Hence there are IMLL proof nets Θ_1' and Θ_2' such that $\Theta_1 \Rightarrow_{\text{Curry}_1} \Theta_1'$ and $\Theta_2 \Rightarrow_{\text{Curry}_1} \Theta_2'$. From Lemma 11, we can see that $\Theta_1' \neq \Theta_2'$. Moreover from Lemma 12 we can find wrapping nets $C_1[]$ and $C_2[]$ such that $C_1[\Theta_1] = \Theta_1'$ and $C_2[\Theta_2] = \Theta_2'$. But since from the proof of Lemma 12 we can see that the way of constructing wrapping nets $C_1[]$ and $C_2[]$ only depends on the positive conclusion of A^+, we can choose a wrapping net $C[]$ such that $C[] = C_1[] = C_2[]$. Then $C[\Theta_1] \neq C[\Theta_2]$.

(2) The case where $\text{pol}(B_1 \otimes B_2 \multimap p, A^+) = +$:

Similar to the case above. □

Corollary 2. *Let Θ_1 and Θ_2 be closed pseudo IIMLL proof nets with the same positive conclusion A^+ such that $\Theta_1 \neq \Theta_2$ and A^+ is an essentially third-order formula. Then there is a wrapping net $C^{A^+}_{A_0^+}[]$ such that A_0^+ is an IIMLL formula with order less than 4 and $C^{A^+}_{A_0^+}[\Theta_1] \neq C^{A^+}_{A_0^+}[\Theta_2]$.*

Proof. If A^+ is an IIMLL formula with order less than 4, then we let A_0^+ be A^+ and $C^{A^+}_{A_0^+}[]$ be the η-expansion of the IMLL proof net exactly consisting of one ID-link with the conclusions A^- and A^+.

Next we consider the case where A^+ is not an IIMLL formula with order less than 4. Then for each normal IMLL proof net Θ, we define $m_{\text{Curry}}(\Theta)$ to be

$$\sum_{L \text{ is a } \otimes^- \text{-link occurrence}} |\{L' \mid L' \text{ is above } L \ \wedge\ L' \text{ is a } \otimes^+\text{-link occurrence}\}|$$
$$+ \sum_{L \text{ is a } \wp^+ \text{-link occurrence}} |\{L' \mid L' \text{ is above } L \ \wedge\ L' \text{ is a } \wp^-\text{-link occurrence}\}|.$$

Moreover we define $m_{\text{Curry}}(\langle \Theta_1, \Theta_2 \rangle)$ to be $m_{\text{Curry}}(\Theta_1) + m_{\text{Curry}}(\Theta_2)$. If we obtain $\langle C[\Theta_1], C[\Theta_2] \rangle$ from $\langle \Theta_1, \Theta_2 \rangle$ applying Proposition 15, it is obvious that $m_{\text{Curry}}(\langle C[\Theta_1], C[\Theta_2] \rangle) < m_{\text{Curry}}(\langle \Theta_1, \Theta_2 \rangle)$. By repeating the procedure, we can obtain a list of wrapping terms $C_1[], \ldots, C_n[]$ such that $C_n[\cdots[C_1[\Theta_1]]\cdots] \neq C_n[\cdots[C_1[\Theta_2]]\cdots]$ and $m_{\text{Curry}}(\langle C_n[\cdots[C_1[\Theta_1]]\cdots], C_n[\cdots[C_1[\Theta_2]]\cdots] \rangle) = 0$. It is obvious that the positive conclusion of $C_n[\cdots[C_1[\Theta_1]]\cdots]$ and $C_n[\cdots[C_1[\Theta_2]]\cdots]$ is an IMLL formula with order less than 4. Moreover applying Lemma 3 repeatedly, we can obtain a wrapping net $C[]$ such that $C[] = C_n[\cdots[C_1[]]\cdots]$. So, $C[\Theta_1] \neq C[\Theta_2]$. □

Theorem 2. *Let Θ_1 and Θ_2 be two IMLL proof nets with the same conclusion A^+ such that $\Theta_1 \neq \Theta_2$. Then there is a wrapping net $C^{A^+}_{A'_+}[]$ such that A'^+ is an IIMLL formula with order less than 4 and $C^{A^+}_{A'^+}[\Theta_1] \neq C^{A^+}_{A'^+}[\Theta_2]$.*

Proof. From Proposition 11, Corollaries 1 and 2, we obtain three wrapping nets $C'^{A^+}_{A_1^+}[]$, $C''^{A_1^+}_{A_2^+}[]$, and $C'''^{A_2^+}_{A'^+}[]$ such that $C'''^{A_2^+}_{A'^+}[C''^{A_1^+}_{A_2^+}[C'^{A^+}_{A_1^+}[\Theta_1]]] \neq C'''^{A_2^+}_{A'^+}[C''^{A_1^+}_{A_2^+}[C'^{A^+}_{A_1^+}[\Theta_2]]]$, where A_1^+ is a pseudo IIMLL formula, A_2^+ an essentially

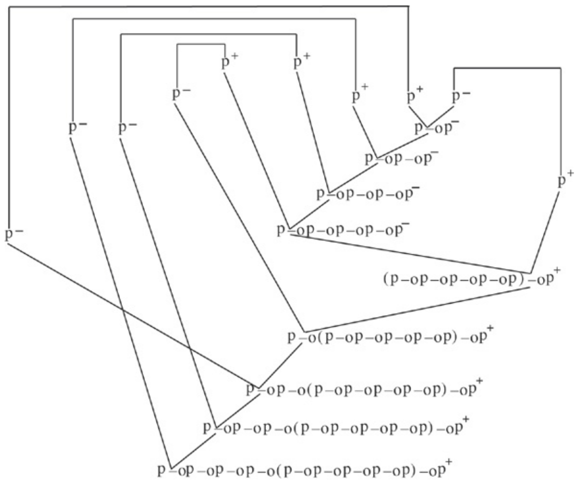


Fig. 32. Θ_1^d.

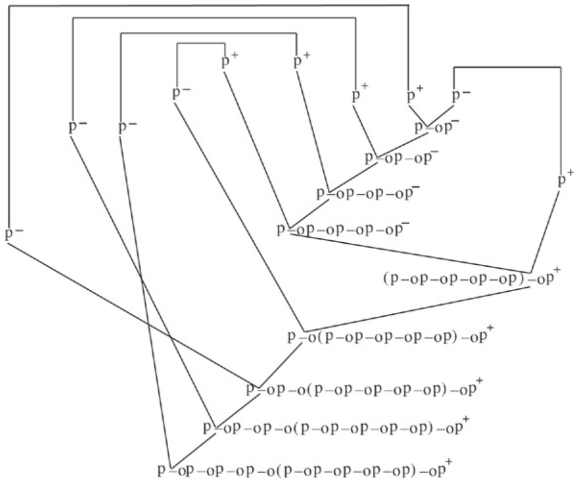


Fig. 33. Θ_2^d.

third-order formula, and A'^{+} is an IIMLL formula with order less than 4. Then from Lemma 3 we can obtain a wrapping net $C^{A^+}_{A'_+}[] = C'''^{A_2^+}_{A'^+}[C''^{A_1^+}_{A_2^+}[C'^{A^+}_{A_1^+}[]]]$. $\square$

Example 5. Figs. 32 and 33 show two proof nets Θ_1^d and Θ_1^d obtained from Θ_1^c and Θ_1^c of Figs. 26 and 27 by applying Curry_1 three times and Curry_3 two times respectively. We note that $\Theta_1^d \neq \Theta_2^d$ and the conclusion of Θ_1^d and Θ_2^d is an IMLL formula with order less than 4. Following Corollary 2 we obtain a wrapping net $C[]$ such that $C[\Theta_1^c] = \Theta_1^d \neq \Theta_2^d = C[\Theta_2^c]$ (but we omit this).

4. The representations on a partial boolean type

In this section we discuss the closed IIMLL proof nets of two third-order IIMLL formulas:

(1) One is **PBool** $\equiv_{\text{def}} p \multimap (p \multimap p) \multimap (p \multimap p) \multimap p$;
(2) The other is **PBool**$'$ $\equiv_{\text{def}} p \multimap p \multimap (p \multimap p \multimap p) \multimap p$.

In particular we mainly discuss **PBool**, since **PBool** has the key property for our separation result, which is that the closed proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ exactly represent all the constant functions and (positive and negative) projections on $\mathbb{BOOL} = \{0, 1\}$. The main purpose of this section is to prove this key property. We call **PBool** and **PBool**′ *partial boolean types* since neither $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ nor $\overbrace{\mathbf{PBool}' \multimap \cdots \multimap \mathbf{PBool}'}^{n} \multimap \mathbf{PBool}'$ has enough closed proof nets to represent all the boolean functions. Our proof of the representation theorem proceeds as follows:

(1) First we characterize the set *PBFT* of all the normal closed proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ for some n $(n \geq 1)$. But in order to save space, we give the characterization in terms of $\beta\eta$-long normal linear λ-terms.
(2) Second we give an inductive characterization of a class *CP* of boolean functions consisting of exactly constant functions and (positive and negative) projections.
(3) Third we prove that the class of boolean functions which the elements of *PBFT* represent is exactly *CP*.

Before proving the theorem, we give a justification about why **PBool** is the best type for our separation result.
First we introduce the following equivalence relation on IIMLL formulas.

Definition 31. A relation $\equiv$ on IIMLL formulas without polarities is the least relation satisfying the following three conditions:

(1) $\equiv$ is an equivalence relation;
(2) $A \multimap (B \multimap p) \equiv B \multimap (A \multimap p)$; and
(3) If $A_1 \equiv A_2$, then $B \multimap A_1 \equiv B \multimap A_2$ and $A_1 \multimap B \equiv A_2 \multimap B$.

A relation $\equiv$ on IIMLL formulas *with polarities* is defined as follows: ${A_1}^+ \equiv {A_2}^+$ and ${A_1}^- \equiv {A_2}^-$ if $A_1 \equiv A_2$.

For example, $(p \multimap (p \multimap p) \multimap p) \multimap (p \multimap p) \multimap p \multimap p^+ \equiv p \multimap ((p \multimap p) \multimap p \multimap p) \multimap (p \multimap p) \multimap p^+$. In the following in this paper, we identify an IIMLL formula with polarities or without polarities with the equivalence class of the IIMLL formula induced by $\equiv$.
Then we can prove the following interesting property about closed proof nets of IIMLL formulas with order less than 4.

Proposition 16. *Let A^+ be an IIMLL formula with order less than 4. We assume that A^+ has a closed IIMLL proof net. Then*

(1) *the number of the closed IIMLL proof nets of A^+ is 1, 2, or equal to or greater than 6.*
(2) *if the number of the closed IIMLL proof nets of A^+ is 2, then A^+ is* **PBool** *or* **PBool**′.

In order to prove the proposition, we need some preliminaries.

Definition 32 (*Depth*). The depth of an IIMLL proof net Θ (denoted by depth(Θ)) is inductively defined as follows:

(1) If the main path of Θ does not include $\otimes^-$-links, then depth(Θ) is 1.
(2) Otherwise, when all the direct subproof nets of Θ are $\Theta_1, \ldots, \Theta_m$, depth($\Theta$) is $\max\{\text{depth}(\Theta_1), \ldots, \text{depth}(\Theta_m)\} + 1$.

Definition 33 (*Maximal Negative Occurrences*). Let A^+ be an IIMLL formula with order less than 4. Then a subformula occurrence B of A^+ is a maximal negative occurrence of A if $\text{pol}(B, A^+) = -$ and there is no subformula occurrence C of A^+ such that B is also a subformula occurrence of C and $\text{pol}(C, A^+) = -$.

For example, let A^+ be $(p_3 \multimap p_2) \multimap {p_1}^+$, where p_1, p_2, and p_3 are difference occurrences of the same formula p. Then although $\text{pol}(p_2, A^+) = -$, p_2 is not a maximal negative occurrence of A^+. But $p_3 \multimap p_2$ is a maximal negative occurrence.

Lemma 13. *Let A^+ be an IIMLL formula with order less than 4. We assume that A^+ has a closed IIMLL proof net. If A^+ has a negative occurrence of $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 1)$, then the number of maximal negative occurrences of p in A^+ is equal to or greater than n. Otherwise, A^+ must have at least one maximal negative occurrence of p.*

Proof. Let Θ be a closed normal IIMLL proof net of A^+. We prove the lemma by induction on depth(Θ).

(1) The case where depth(Θ) = 1:
In this case Θ has exactly one ID-link that has the conclusions p^- and p^+. Θ does not include any negative occurrence of $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 1)$. Then A^+ must be $p \multimap p$. Then A^+ has exactly one maximal negative occurrence of p.

(2) The case where depth(Θ) > 1:

(a) The case where the negative occurrence $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 1)$ to which we pay attention occurs in the main path of Θ:

Then $\mathbf{DSP}_\Theta$ has at least n elements that are disjoint each other and maximal. Let Θ'_i $(1 \leq i \leq n)$ be such nets. Since A^+ has an order less than 4, the positive conclusion of Θ'_i must be p^+. Then let Θ'_{ic} $(1 \leq i \leq n)$ be a closed net obtained from Θ'_i adding some $\wp^+$-links. Moreover let A'^+_{ic} be the positive conclusion of Θ'_{ic} for each i $(1 \leq i \leq n)$. Then A'^+_{ic} must have an order less than 4 for each i $(1 \leq i \leq n)$, since A^+ has an order less than 4. Hence since depth(Θ'_{ic}) < depth(Θ), by inductive hypothesis, the number of maximal negative occurrences of p in A'^+_{ic} must be equal to or greater than 1, regardless of whether A'^+_{ic} includes a negative occurrence of $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 1)$ for each i $(1 \leq i \leq n)$. Then since the positive conclusion of Θ'_i is p^+, such maximal negative occurrences of p in A'^+_{ic} must be maximal negative occurrences of p in A^+ for each i $(1 \leq i \leq n)$. Therefore the number of maximal negative occurrences of p in A^+ must be equal to or greater than n.

(b) Otherwise:

Then the negative occurrence $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 1)$ to which we pay attention occurs in the main path of an element Θ' of $\mathbf{DSP}_\Theta$ such that Θ' is not Θ. Then let Θ'_c be a closed net obtained from Θ' adding some $\wp^+$-links. Moreover let A'^+_c be the positive conclusion of Θ'_c. Since depth(Θ'_c) < depth(Θ), by inductive hypothesis, the number of maximal negative occurrences of p in A'^+_c must be equal to or greater than n. Then since the positive conclusion of Θ' is p^+, such maximal negative occurrences of p in A'^+_c must be maximal negative occurrences of p in A^+. Therefore the number of maximal negative occurrences of p in A^+ must be equal to or greater than n. □

Proof of Proposition 16. We assume that A^+ is an IIMLL formula with order less than 4 and that it has a closed IIMLL proof net. Then we prove that if A^+ is neither **PBool** nor **PBool**′, then the number of the closed proof nets of A^+ is 1 or equal to or greater than 6. By proving the statement above we can prove (1) and (2) at the same time.

(1) The case where any maximal negative occurrence of $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 1)$ never occurs in A^+.
The only possibility is $A^+ = p \multimap p^+$. In this case, A^+ has the only one closed proof net.

(2) The case where a maximal negative occurrence of $\overbrace{p \multimap \cdots \multimap p}^{n} \multimap p$ $(n \geq 3)$ occurs in A^+.
From Lemma 13, the number of maximal negative occurrences of p is equal to or greater than 3. It is easily seen that the number of closed proof nets of A^+ is equal to or greater than 6, since the number of the combinations that such maximal negative occurrences of p choose $\wp^+$-links is equal to or greater than $3! = 6$.

(3) The case where a maximal negative occurrences of $p \multimap p \multimap p$ occurs in A^+ three times.
It is easily seen that the number of closed proof nets of A^+ is equal to or greater than 6, since the number of the combinations that such maximal negative occurrences $p \multimap p \multimap p$ choose $\wp^+$-links is equal to or greater than $3! = 6$.

(4) The case where a maximal negative occurrences of $p \multimap p$ occurs in A^+ three times.
Similar to the case above.

(5) Otherwise:

(a) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly two respectively:

In this case A^+ must be

$p \multimap p \multimap p \multimap (p \multimap p) \multimap (p \multimap p) \multimap (p \multimap p \multimap p) \multimap (p \multimap p \multimap p) \multimap p^+$.

The number of closed normal proof nets of A^+ is $30 \cdot 2 \cdot 2 \cdot 3! = 6! = 720$.

(b) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly two and one respectively:

In this case A^+ must be

$p \multimap p \multimap p \multimap (p \multimap p) \multimap (p \multimap p \multimap p) \multimap (p \multimap p \multimap p) \multimap p^+$.

The number of closed normal proof nets of A^+ is $10 \cdot 2 \cdot 3! = 5! = 120$.

(c) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly one and two respectively:

In this case A^+ must be

$p \multimap p \multimap (p \multimap p) \multimap (p \multimap p) \multimap (p \multimap p \multimap p) \multimap p^+$.

The number of closed normal proof nets of A^+ is $6 \cdot 2 \cdot 2 = 4! = 24$.

(d) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly one respectively:

In this case A^+ must be

$p \multimap p \multimap (p \multimap p) \multimap (p \multimap p \multimap p) \multimap p^+$.

The number of closed normal proof nets of A^+ is $3 \cdot 2 = 3! = 6$.

(e) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly two and zero respectively:

In this case A^+ must be

$p \multimap p \multimap p \multimap (p \multimap p \multimap p) \multimap (p \multimap p \multimap p) \multimap p^+$.

The number of closed normal proof nets of A^+ is $2 \cdot 3! \cdot 2 = 4! = 24$.

(f) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly one and zero respectively:

In this case A^+ must be **PBool**$'$. So this case is not possible.

(g) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly zero and two respectively:

In this case A^+ must be **PBool**. So this case is not possible.

(h) The case where both the number of the maximal negative occurrences of $p \multimap p \multimap p$ and that of $p \multimap p$ are exactly zero and one respectively:

In this case A^+ must be $p \multimap (p \multimap p) \multimap p$. The number of A^+ is exactly one. □

By Proposition 16 candidates are restricted to **PBool** and **PBool**$'$. However while closed proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ can represent constant functions and projections, which is the key property in order to establish our separation result, closed proof nets of $\overbrace{\mathbf{PBool}' \multimap \cdots \multimap \mathbf{PBool}'}^{n} \multimap \mathbf{PBool}'$ can only represent parity check functions: we can only judge whether the number of the occurrences of 1 (or 0) of a given sequence with n bits is odd or even. So the only remaining candidate is **PBool**. That is why we choose **PBool**. In what follows in this section, we mainly concentrate on proving the key property.

4.1. A characterization of $PBFT^n$

In this subsection we give a characterization of the closed normal IIMLL proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$. At the beginning we introduce a linear λ-term assignment system to normal IIMLL proof nets. The reason why we introduce this system is purely to save space. We could do the same thing in terms of normal IIMLL proof nets.

Fig. 34 shows the term assignment system. Using the system we assign the canonical derivation to each normal IIMLL proof net.

(1) ID-axiom:

$$\overline{x : p^-,\ x : p^+}$$

where to each ID-axiom a different variable x is assigned.

(2) $\otimes^-$-rule:

$$\frac{x_1 : A_1^-, \ldots, x_m : A_m^-, t : A^+ \qquad y : B^-, y_1 : B_1^-, \ldots, y_n : B_n^-, u : C^+}{x_1 : A_1^-, \ldots, x_m : A_m^-, z : A \multimap B^-, y_1 : B_1^-, \ldots, y_n : B_n^-, u\langle z\, t/y\rangle : C^+}$$

where z is a new variable.

(3) $\wp^+$-rule

$$\frac{x : A^-, x_1 : A_1^-, \ldots, x_m : A_m^-, u : B^+}{x_1 : A_1^-, \ldots, x_m : A_m^-, \lambda x.u : A \multimap B^+}$$

Fig. 34. A linear λ-term-assignment system.

Definition 34 (*Canonical Derivations*). Let Θ be a normal IIMLL proof net. Then we define the canonical derivation $\mathrm{deriv}(\Theta)$ of Θ inductively on the depth of Θ:

(1) The case where $\mathrm{depth}(\Theta) = 1$:

If Θ consists of exactly one ID-link with the conclusions p^- and p^+, then ID-axiom $\overline{x{:}p^-, x{:}p^+}$ is $\mathrm{deriv}(\Theta)$.

Otherwise, Θ consists of exactly one ID-link with the conclusions p^-, p^+ and exactly one $\wp^+$-link with the conclusion $p \multimap p^+$. Then $\mathrm{deriv}(\Theta)$ is the derivation obtained by applying $\wp^+$-rule to the conclusions p^- and p^+ of an ID-axiom.

(2) The case where $\mathrm{depth}(\Theta) > 1$:

Let the positive conclusion of Θ be $B_1 \multimap \cdots \multimap B_\ell \multimap p^+$. Then there is the list $\Theta^1, \ldots, \Theta^m$ of the direct subproof nets of Θ such that

(a) each Θ^j has the positive conclusion ${A_j}^+$ for each j $(1 \le j \le m)$;

(b) Θ has a maximal negative formula occurrence $A_1 \multimap \cdots \multimap A_m \multimap p^-$ such that each subformula occurrence $A_j \multimap \cdots \multimap A_m \multimap p^-$ and ${A_j}^+$ are the conclusion and the left premise of a $\otimes^-$-link for each j $(1 \le j \le m)$.

Then $\mathrm{deriv}(\Theta)$ is obtained by the following steps:

(a) We apply $\otimes^-$-rule to an ID-axiom with the conclusions p^- and p^+ and $\mathrm{deriv}(\Theta_m)$ and obtain a derivation Π_m with a conclusion $A_m \multimap p^-$.

(b) We apply $\otimes^-$-rule to Π_j and $\mathrm{deriv}(\Theta_{j-1})$ and obtain a derivation Π_{j-1} with a conclusion $A_{j-1} \multimap \cdots \multimap A_m \multimap p^-$ for each j $(2 \le j \le m)$.

(c) We apply $\wp^+$-rule to the derivation Π_1 obtaining a derivation with the positive conclusion $B_\ell \multimap p^+$. Let the derivation be Λ_ℓ.

(d) We apply $\wp^+$-rule to the derivation Λ_k obtaining a derivation Λ_{k-1} with the positive conclusion $B_{k-1} \multimap \cdots \multimap B_\ell \multimap p^+$ for each k $(2 \le k \le \ell)$.

(e) Let Λ_1 be $\mathrm{deriv}(\Theta)$.

Let the linear λ-term typed by the positive conclusion of $\mathrm{deriv}(\Theta)$ be $\mathrm{term}(\Theta)$.

Lemma 14. *Let Θ be a normal IIMLL proof net. Then* $\mathrm{term}(\Theta)$ *has any of the following forms:*

(1) $\mathrm{term}(\Theta) = x$ *for some variable* x *or* $\mathrm{term}(\Theta) = \lambda x.x$.

(2) $\mathrm{term}(\Theta) = \lambda x_1.\ldots\lambda x_n.(\cdots(x\ \mathrm{term}(\Theta^1))\cdots)\,\mathrm{term}(\Theta^m)$, *where* $\{\Theta_1, \ldots, \Theta_m\}$ *is the set of all direct subproof nets of* Θ *and* x *may or may not belong to* $\{x_1, \ldots, x_n\}$.

Proof. Induction on the depth of Θ.

(1) The case where depth$(\Theta) = 1$:

Then deriv(Θ) consists of exactly one ID-link or exactly one ID-link and one $\wp^+$-link. In this case by definition, term$(\Theta) = x$ for some variable x or term$(\Theta) = \lambda x.x$.

(2) Otherwise:

Then we can obtain deriv(Θ) from the set $\{\Theta^1, \ldots, \Theta^m\}$ of the direct subproof nets of Θ following Definition 34. Since term(Θ^i) is the linear typed λ-term typed by the positive conclusion of deriv(Θ^i), we can easily see term$(\Theta) = \lambda x_1 \ldots . \lambda x_n.(\cdots(x \text{ term}(\Theta^1))\cdots) \text{term}(\Theta^m)$. □

Then the following proposition holds.

Proposition 17. *If Θ_1 and Θ_2 are normal IMLL proof nets with the same positive conclusion such that $\Theta_1 \neq \Theta_2$, then* term$(\Theta_1) \not\equiv_\alpha$ term$(\Theta_2)[y'_1/y_1, \ldots, y'_k/y_k]$ *for any substitution* $[y'_1/y_1, \ldots, y'_k/y_k]$ *such that* $y'_j (1 \leq j \leq k)$ *occurs once exactly, where* $\equiv_\alpha$ *is the α-congruence in the usual λ-calculus theory [1].*

Proof. We prove that if term$(\Theta_1) \equiv_\alpha$ term$(\Theta_2)[y'_1/y_1, \ldots, y'_k/y_k]$ for a substitution $[y'_1/y_1, \ldots, y'_k/y_k]$ such that $y'_j (1 \leq j \leq k)$ occurs once exactly, then $\Theta_1 = \Theta_2$ by induction on the depth of Θ_1.

(1) The case where depth$(\Theta_1) = 1$:

(a) The case where term$(\Theta_1) \equiv x \equiv$ term$(\Theta_2)[x/y]$:

Then since both Θ_1 and Θ_2 are an IIMLL proof net consisting of exactly one ID-link with the conclusions p^- and p^+, it is obvious that $\Theta_1 = \Theta_2$.

(b) The case where term$(\Theta_1) \equiv \lambda x.x \equiv$ term(Θ_2):

Then since both Θ_1 and Θ_2 are an IIMLL proof net consisting of exactly one ID-link with the conclusions p^- and p^+ and exactly one $\wp^+$-link with the positive conclusion $p \multimap p^+$, it is obvious that $\Theta_1 = \Theta_2$.

(2) The case where depth$(\Theta_1) > 1$:

Then by definition, there are m $(m \geq 1)$ and a set of IIMLL proof nets $\{\Theta_1^1, \ldots, \Theta_1^m\}$ that is the set of all direct subproof net of Θ_1. Then from Lemma 14 term(Θ_1) has the form $\lambda x_1 \ldots . \lambda x_n.(\cdots(x \text{ term}(\Theta_1^1))\cdots) \text{term}(\Theta_1^m)$, where $n \geq 0$ and x may or may not belong to $\{x_1, \ldots, x_n\}$. Since term$(\Theta_1) \equiv_\alpha$ term$(\Theta_2)[y'_1/y_1, \ldots, y'_k/y_k]$, term$(\Theta_2)[y'_1/y_1, \ldots, y'_k/y_k] \equiv_\alpha \lambda x_1 \ldots . \lambda x_n.(\cdots(x \text{ term}(\Theta_1^1))\cdots) \text{term}(\Theta_1^m)$. On the other hand, depth$(\Theta_2) > 1$ because if not, then term(Θ_2) would be a variable or $\lambda z.z$ for some variable z. So, when we let θ be $[y'_1/y_1, \ldots, y'_k/y_k]$, from Lemma 14 term$(\Theta_2)\,\theta$ must have the form

$$\lambda x'_1 \ldots . \lambda x'_{n'}.(\cdots((x'\,\theta)\,(\text{term}(\Theta_2^1)\,\theta))\cdots)(\text{term}(\Theta_2^{m'})\,\theta),$$

where $m' \geq 1$, $n' \geq 0$ and x' may or may not belong to $\{x'_1, \ldots, x'_{n'}\}$ and $\Theta_2^{i'}$ is a direct subproof net of Θ_2 for each i' $(1 \leq i' \leq m')$. Then $n = n'$, $m = m'$, $x = x'[y'_1/y_1, \ldots, y'_k/y_k, x_1/x'_1, \ldots, x_n/x'_n]$, and

$$\text{term}(\Theta_1^i) \equiv_\alpha \text{term}(\Theta_2^i)[y'_1/y_1, \ldots, y'_k/y_k, x_1/x'_1, \ldots, x_n/x'_n]$$

for each i $(1 \leq i \leq m)$. Then by inductive hypothesis $\Theta_1^i = \Theta_2^i$ for each i $(1 \leq i \leq m)$. Hence $\Theta_1 = \Theta_2$. □

From the proposition above we can identify a normal IIMLL proof net Θ with term(Θ).

Next we consider the closed normal linear λ-terms assigned to **PBool**. Fig. 35 shows two closed proof nets $\underline{0}$ and $\underline{1}$ of **PBool**. While the linear λ-term $\lambda x.\lambda f.\lambda g.g(fx)$ corresponds to the IIMLL proof net $\underline{0}$, $\lambda x.\lambda f.\lambda g.f(gx)$ corresponds to $\underline{1}$.

4.1.1. The closed normal terms on **PBool** $\multimap$ **PBool**

Next we classify the closed $\beta\eta$-long normal terms of **PBool** $\multimap$ **PBool** as a preliminary step. This subsection is in fact redundant, but seems useful for understanding the general case. Since the closed $\beta\eta$-long normal terms on the formula have always the form $\lambda F.\lambda x.\lambda f.\lambda g.t$, we only write down the body t instead of writing down the whole term in the following.

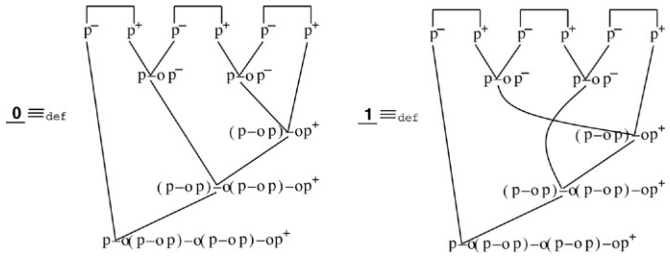


Fig. 35. $\underline{0}$ and $\underline{1}$.

We classify them according to the surrounding contexts of f and g.

(a) The case where f and g occurs with the form $f(g(t'))$ or $g(f(t'))$:
(1) $Fx(\lambda y_1.y_1)(\lambda y_2.f(gy_2))$ and (2) $Fx(\lambda y_1.y_1)(\lambda y_2.g(fy_2))$ and (3) $Fx(\lambda y_1.f(gy_1))(\lambda y_2.y_2)$ and (4) $Fx(\lambda y_1.g(fy_1))(\lambda y_2.y_2)$ and (5) $f(g(Fx(\lambda y_1.y_1)(\lambda y_2.y_2)))$ and (6) $g(f(Fx(\lambda y_1.y_1)(\lambda y_2.y_2)))$ and (7) $F(f(gx))(\lambda y_1.y_1)(\lambda y_2.y_2)$ and (8) $F(g(fx))(\lambda y_1.y_1)(\lambda y_2.y_2)$

(b) Otherwise:
(9) $Fx(\lambda y_1.fy_1)(\lambda y_2.gy_2)$ and (10) $Fx(\lambda y_1.gy_1)(\lambda y_2.fy_2)$. While the first term denotes the identity function on $\{\underline{0}, \underline{1}\}$, the second term the negation. The terms of the other cases are a constant function on $\{\underline{0}, \underline{1}\}$. Note that in order for a term to denote a non-constant function, in the term, f and g must occur in the second argument and the third argument of F separately, because for F, $\lambda x.\lambda f.\lambda g.g(fx)$ or $\lambda x.\lambda f.\lambda g.f(gx)$ is substituted. (11) $f(F(gx)(\lambda y_1.y_1)(\lambda y_2.y_2))$ and (12) $g(F(fx)(\lambda y_1.y_1)(\lambda y_2.y_2))$ and (13) $f(Fx(\lambda y_1.y_1)(\lambda y_2.gy_2))$ and (14) $g(Fx(\lambda y_1.y_1)(\lambda y_2.fy_2))$ and (15) $f(Fx(\lambda y_1.gy_1)(\lambda y_2.y_2))$ and (16) $g(Fx(\lambda y_1.fy_1)(\lambda y_2.y_2))$ and (17) $F(fx)(\lambda y_1.y_1)(\lambda y_2.gy_2)$ and (18) $F(gx)(\lambda y_1.y_1)(\lambda y_2.fy_2)$ and (19) $F(fx)(\lambda y_1.gy_1)(\lambda y_2.y_2)$ and (20) $F(gx)(\lambda y_1.fy_1)(\lambda y_2.y_2)$.

4.1.2. *The general case*

Next we give a characterization of the closed normal terms of $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$. First we note that the closed $\beta\eta$-long normal terms on the formula always have the form $\lambda F_1.\cdots\lambda F_n.\lambda x.\lambda f.\lambda g.t'$. We define a set *PrePBFT* of λ-terms including such bodies t' inductively.

(1) $x \in \mathit{PrePBFT}$;
(2) If $t' \in \mathit{PrePBFT}$, then $f\,t', g\,t' \in \mathit{PrePBFT}$;
(3) If $t'_1, t'_2, t'_3 \in \mathit{PrePBFT}$, then $F_i\, t'_1\, (\lambda x.t'_2)\, (\lambda x.t'_3)$.

Then we easily see that by induction all the λ-bindings of each element $t \in \mathit{PrePBFT}$ are a linear binding and x occurs exactly once in t as a free variable. Then we define

$$\begin{aligned} \mathit{PBFT}^n \equiv_{\mathrm{def}} \{ & \lambda F_1.\dots.\lambda F_n.\lambda x.\lambda f.\lambda g.t' \mid t' \in \mathit{PrePBFT} \\ & \wedge FV(t') = \{F_1, \dots, F_n, f, g, x\} \\ & \wedge (F_1, \dots, F_n, f, g, \text{ and } x \text{ occur in } t' \text{ exactly once})\} \end{aligned}$$

for $n \geq 1$. Moreover we define $\mathit{PBFT} \equiv_{\mathrm{def}} \bigcup_{n\geq 1} \mathit{PBFT}^n$.

Proposition 18. *PBFT^n is the set of all closed normal terms of* $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$.

Proof. Let CNT^n be the set of all closed normal terms of $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$.

(1) $\mathit{CNT}^n \subseteq \mathit{PBFT}^n$:
Let $t = \lambda F_1.\dots.\lambda F_n.\lambda x.\lambda f.\lambda g.t'$ be in CNT^n. Then we define $\mathit{Subterm}_p(t') = \{s' : p \mid s' \text{ is a subterm of } t'\}$. We easily see that $s' \in \mathit{Subterm}_p(t')$ iff

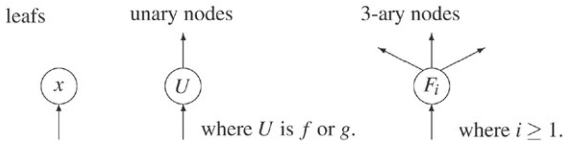


Fig. 36. Nodes for *PrePBFTree*.

(a) $s' = x$,
(b) $s' = f\,u'$ or $s' = g\,u'$, where $u' \in \mathit{Subterm}_p(t')$, or
(c) $s' = F_i\,t'_1\,(\lambda x.t'_2)(\lambda x.t'_3)$ for some i $(1 \le i \le n)$, where $t'_1, t'_2, t'_2 \in \mathit{Subterm}_p(t')$.
Hence $t' \in \mathit{Subterm}_p(t') \subseteq \mathit{PrePBFT}$. Moreover $FV(t') = \{F_1, \dots, F_n, f, g, x\}$, and $F_1, \dots, F_n, f, g$, and x occur exactly once in t'. So $t \in \mathit{PBFT}^n$.

(2) $\mathit{PBFT}^n \subseteq \mathit{CNT}^n$:
Let $t = \lambda F_1.\dots.\lambda F_n.\lambda x.\lambda f.\lambda g.t'$ be in PBFT^n. Then it is obvious that t is normal and t is typed by $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$. So $t \in \mathit{CNT}^n$. $\square$

4.2. *The counting of the normal closed proof nets of* $\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool} \multimap \mathbf{PBool}$

In this subsection, we present the recursive equations which give the number of the normal closed proof nets of $\overbrace{\mathbf{PBool} \multimap \cdots \multimap \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$. In order to do that, we define the set of trees *PBFTree* that is isomorphic to *PBFT*.

Definition 35 (*PrePBFTree*). A set *PrePBFTree* is the set of trees consisting of nodes shown in Fig. 36, where three outgoing edges of a node with label F_i are distinguished. $\mathit{PBFTree}^n$ is the set of elements of *PrePBFTree* satisfying the following conditions:

(1) the number of occurrences of 3-ary nodes is n and the labels are $F_1, \dots, F_n$, and
(2) unary nodes occur exactly twice, one node has the label f, and the other g.

Finally, we define $\mathit{PBFTree} \equiv_{\mathrm{def}} \bigcup_{n \ge 1} \mathit{PBFTree}^n$.

Proposition 19. *There is a bijection between PrePBFT and PrePBFTree.*

Proof. Let *SubPrePBFT* be the set of all subterms of *PrePBFT* and *SubPrePBFTree* be the set of all subtrees of *PrePBFTree*. Fig. 37 shows that a function $(-)^* : \mathit{SubPrePBFT} \to \mathit{SubPrePBFTree}$ is defined inductively. Moreover Fig. 38 shows that a function $(-)^{\mathrm{rev}} : \mathit{SubPrePBFTree} \to \mathit{SubPrePBFT}$ is defined inductively.

Claim 5. *If $t \in \mathit{SubPrePBFT}$ is not a λ-abstraction, then $((t)^*)^{\mathrm{rev}} = t$.*

Proof of Claim 5. We prove this claim by induction on the structure of t.

(1) The case where $t = x$:
It is obvious that $((x)^*)^{\mathrm{rev}} = x$.
(2) The case where $t = f\,t'$ or $t = g\,t'$:
Then t' is not λ-abstraction. So by inductive hypothesis, $((t')^*)^{\mathrm{rev}} = t'$. It is obvious that $((f\,t')^*)^{\mathrm{rev}} = f((t')^*)^{\mathrm{rev}} = f\,t'$ and $((g\,t')^*)^{\mathrm{rev}} = g((t')^*)^{\mathrm{rev}} = g\,t'$.
(3) The case where $t = F_i\,t_1\,t_2\,t_3$:
Then t_2 and t_3 have the forms $\lambda x.t'_2$ and $\lambda x.t'_3$ respectively, where t'_2 and t'_3 are not λ-abstraction. Then by inductive hypothesis, $((t'_2)^*)^{\mathrm{rev}} = t'_2$ and $((t'_3)^*)^{\mathrm{rev}} = t'_3$. Since t_1 is not λ-abstraction, by inductive hypothesis, $((t_1)^*)^{\mathrm{rev}} = t_1$. Using these three equations we can show that

$$\begin{aligned}((F_i\,t_1\,t_2\,t_3)^*)^{\mathrm{rev}} &= F_i\,((t_1)^*)^{\mathrm{rev}}\,(\lambda x.((t'_2)^*)^{\mathrm{rev}})\,(\lambda x.((t'_3)^*)^{\mathrm{rev}}) \\ &= F_i\,t_1\,(\lambda x.t'_2)\,(\lambda x.t'_3) = F_i\,t_1\,t_2\,t_3. \quad \square\end{aligned}$$

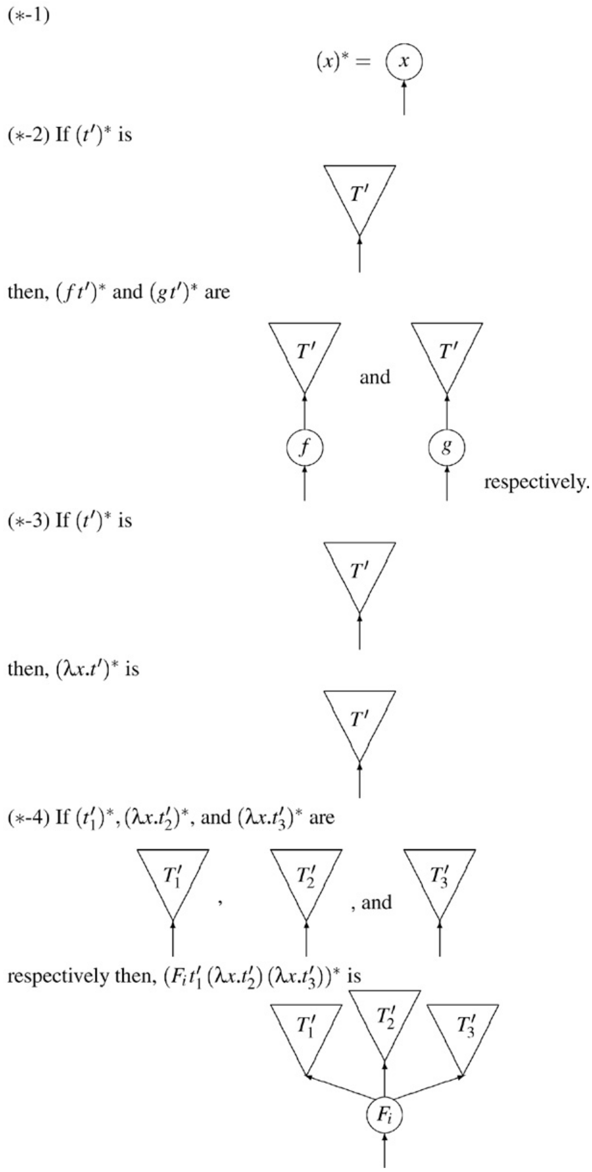
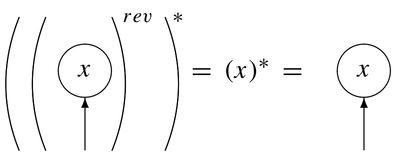


Fig. 37. The definition of $(-)^*$.

Claim 6. *If $T \in$ SubPrePBFTree, then $((T)^{\mathrm{rev}})^* = T$.*

Proof of Claim 6. (1)



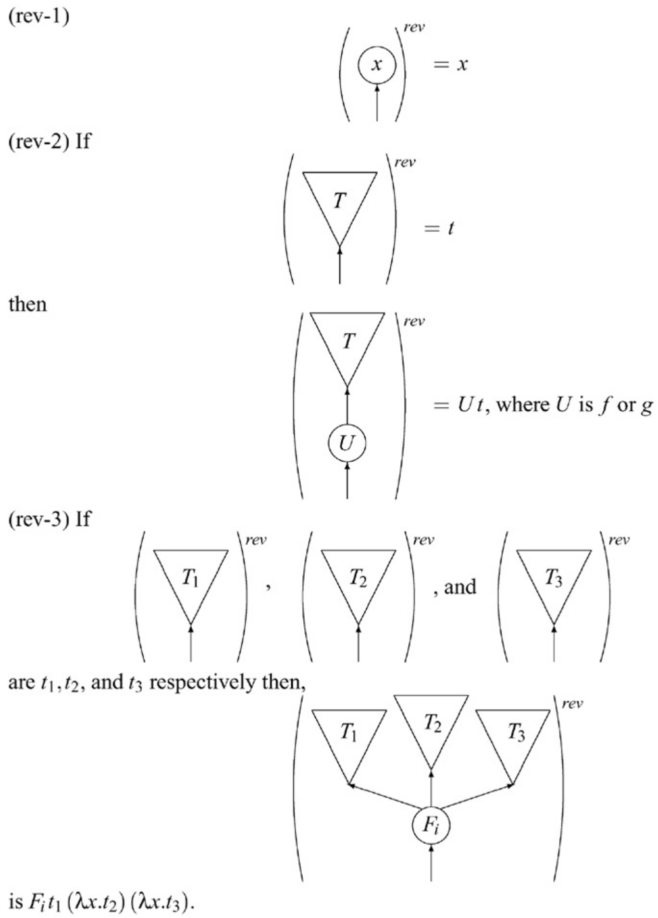
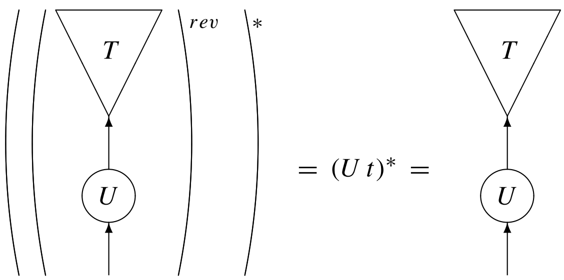
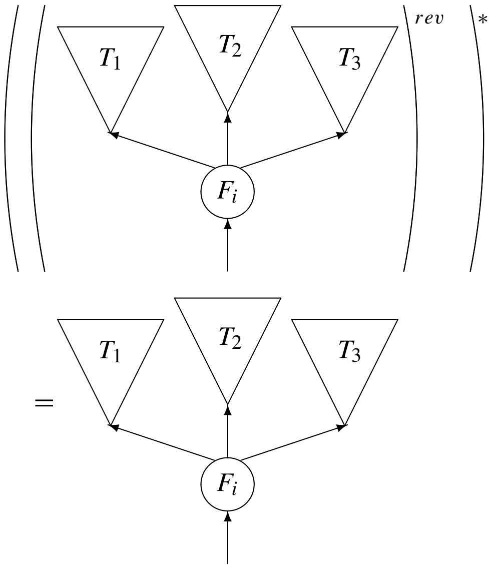


Fig. 38. The definition of $(-)^{\text{rev}}$.

(2)



where U is f or g, $(T)^{\text{rev}} = t$, and we use $(t)^* = ((T)^{\text{rev}})^* = T$, which is derived from inductive hypothesis.
(3)



$= (F\ t_1\ (\lambda x.t_2)\ (\lambda x.t_3))^*$

where $(T_1)^{\text{rev}} = t_1$, $(T_2)^{\text{rev}} = t_2$,and $(T_3)^{\text{rev}} = t_3$, and we use $(t_1)^* = ((T_1)^{\text{rev}})^* = T_1$, $(\lambda x.t_2)^* = (t_2)^* = ((T_2)^{\text{rev}})^* = T_2$, and $(\lambda x.t_3)^* = (t_3)^* = ((T_3)^{\text{rev}})^* = T_3$, which are derived from inductive hypothesis. $\square$

Then when we consider the restrictions $(-)^*|_{PrePBFT}$ of $(-)^*$ to *PrePBFT* and $(-)^{\text{rev}}|_{PrePBFTree}$ of $(-)^{\text{rev}}$ to *PrePBFTree*, we can easily see that $(-)^*|_{PrePBFT}$ is a function from *PrePBFT* to *PrePBFTree*, $(-)^{\text{rev}}|_{PrePBFTree}$ is a function from *PrePBFTree* to *PrePBFT*, and $(-)^{\text{rev}}|_{PrePBFTree}$ is the inverse of $(-)^*|_{PrePBFT}$. $\square$

Corollary 3. *There is a bijection between $PBFT^n$ and $PBFTree^n$.*

Proof. We define the map $\text{strip}^n : PBFT^n \to PrePBFT$ by $\lambda F_1. \ldots .\lambda F_n.\lambda x.\lambda f.\lambda g.t' \mapsto t'$. It is obvious that strip^n is injective. Hence the bijection $(\text{strip}^n)^{-1} : \text{strip}^n(PBFT^n) \to PBFT^n$ can be defined. Next we consider the maps $(-)^*|_{PrePBFT} \circ \text{strip}^n$ and $(\text{strip}^n)^{-1} \circ (-)^{\text{rev}}|_{PBFTree^n}$. We can easily show that $(-)^*|_{PrePBFT} \circ \text{strip}^n$ maps an element of $PBFT^n$ into an element of $PBFTree^n$ by induction on the structure of terms. So $(-)^*|_{PrePBFT} \circ \text{strip}^n$ is a function from $PBFT^n$ to $PBFTree^n$. Similarly we can easily show that $(\text{strip}^n)^{-1}$ and $(-)^{\text{rev}}|_{PBFTree^n}$ are composable, i.e., $\text{cod}((-)^{\text{rev}}|_{PBFTree^n}) = \text{strip}^n(PBFT^n) = \text{dom}((\text{strip}^n)^{-1})$. Moreover it is obvious that $(\text{strip}^n)^{-1} \circ (-)^{\text{rev}}|_{PBFTree^n}$ is the inverse of $(-)^*|_{PrePBFT} \circ \text{strip}^n$, because $PBFTree^n$ is a subset of *PrePBFTree*. $\square$

By Corollary 3 we have established the reduction of the counting problem of $PBFT^n$ to that of $PBFTree^n$. In the following we discuss the counting problem of $PBFTree^n$. Moreover we reduce the problem to the following three subproblems:

(1) We count the set $3AryTr^n$ whose elements are a tree consisting of only n 3-ary nodes $\overbrace{F, \ldots, F}^{n}$ and leafs, i.e., the set of trees obtained from $PBFTree^n$, making the labels of 3-ary nodes indistinguishable and removing any unary nodes.
(2) Then we count the combinations of possible occurrences of two unary nodes labeled by f and g in $3AryTr^n$.
(3) Finally we must consider that the labels of 3-ary nodes in $3AryTr^n$ are different from each other in $PBFTree^n$. This is simply to multiply the number obtained above by $n!$.

4.2.1. The cardinality of $|3AryTr^n|$

(1) The case where $n = 1$:
In this case since the only 3-ary node only occurs as the root, $|3AryTr^1| = 1$.
(2) The case where $n = 2$:
In this case since two 3-ary nodes occur as the root and a son of the root, $|3AryTr^2| = 3$.
(3) The case where $n = 3$:

(a) The case where exactly two sons of the root are leafs:

In this case, each 3-ary node has a different height. Then the number of the combinations is $\binom{3}{1} \times \binom{3}{1} = 3 \times 3 = 9$.

(b) The case where exactly one son is a leaf:

In this case, one 3-ary node occurs as the root and the others occur as a son of the root. Then the number of the combinations is $\binom{3}{2} = 3$.

So, $|3AryTr^3| = 9 + 3 = 12$.

(4) The case where $n \geq 4$:

(a) The case where exactly two sons of the root are leafs:

In this case the number of the combinations is $\binom{3}{1} \cdot |3AryTr^{n-1}|$.

(b) The case where exactly one sons of the root is a leaf:

In this case the number of the combinations is

$$\binom{3}{2} \cdot \sum_{k+\ell=n-1,\, k,l \geq 1} |3AryTr^k| \cdot |3AryTr^\ell|.$$

(c) The case where each son of the root is not a leaf:

In this case the number of the combinations is

$$\binom{3}{3} \cdot \sum_{k+\ell+m=n-1,\, k,l,m \geq 1} |3AryTr^k| \cdot |3AryTr^\ell| \cdot |3AryTr^m|.$$

Summing up, we obtain

$$\begin{aligned} |3AryTr^n| \;=\; & \binom{3}{1} \cdot |3AryTr^{n-1}| \\ & + \binom{3}{2} \cdot \sum_{k+\ell=n-1,\, k,l \geq 1} |3AryTr^k| \cdot |3AryTr^\ell| \\ & + \binom{3}{3} \cdot \sum_{k+\ell+m=n-1,\, k,l,m \geq 1} |3AryTr^k| \cdot |3AryTr^\ell| \cdot |3AryTr^m|. \end{aligned}$$

4.2.2. The combinations of insertions of two unary nodes labeled by f and g respectively in $3AryTr^n$

To count the combinations, we note the following property about $3AryTr^n$.

Proposition 20. *Each element of $3AryTr^n$ has the same number of edges. The number is $3n$.*

Proof. By induction on n.

(1) The case where $n = 1$:

The only element of $3AryTr^1$ has three edges.

(2) The case where $n = 2$:

All the elements of $3AryTr^2$ have six edges.

(3) The case where $n = 3$:

All the elements of $3AryTr^3$ have nine edges.

(4) The case where $n \geq 4$:

(a) The case where exactly two sons of the root are leafs:

(b) The case where exactly one sons of the root is a leaf:

(c) The case where each son of the root is not a leaf:

In any of these three cases, we can easily see that each element of $3AryTr^n$ has $3n$ edges. □

From Proposition 20 we can see that for each element T of $3AryTr^n$ the combinations to insert exactly two nodes labeled by f and g to T is that to put f, g, $f\,g$, or $g\,f$ on $3n + 1$ places such that f and g occur exactly once respectively. Since we must consider possibilities of putting f, g, $f\,g$, or $g\,f$ on the place immediately before of the root of T, the number of the places is $3n + 1$, not $3n$. We count the combinations by case analysis:

(1) The case where f and g are put on separately:
In this case the number of the combinations is $2 \cdot \binom{3n+1}{2}$.
(2) The case where f and g are put as $f\ g$ or $g\ f$:
In this case the number of the combinations is $2 \cdot (3n + 1)$.

Summing up, we obtain the total number of the combinations for each element T of $3AryTr^n$ is $2 \cdot \left(\binom{3n+1}{2} + 3n + 1\right)$ for each n $(n \geq 1)$.

4.2.3. The cardinality $|PBFTree^n|$

As we said before, we must consider that the label F of 3-ary nodes in $3AryTr^n$ are replaced by pairwisely different labels $F_1, \ldots, F_n$ in $PBFTree^n$. This is simply to multiply the number obtained above by $n!$. Then for each $n \geq 1$, the cardinality $|PBFTree^n|$ is given as follows:

(1) $|PBFTree^1| = 1! \cdot \left(2 \cdot \left(\binom{3+1}{2} + 3 + 1\right)\right) \cdot |3AryTr^1| = 20$.
(2) $|PBFTree^2| = 2! \cdot \left(2 \cdot \left(\binom{3 \cdot 2+1}{2} + 3 \cdot 2 + 1\right)\right) \cdot |3AryTr^2| = 336$.
(3) $|PBFTree^3| = 3! \cdot \left(2 \cdot \left(\binom{3 \cdot 3+1}{2} + 3 \cdot 3 + 1\right)\right) \cdot |3AryTr^3| = 7920$.
(4) The case where $n \geq 4$:
$|PBFTree^n| = n! \cdot \left(2 \cdot \left(\binom{3 \cdot n+1}{2} + 3 \cdot n + 1\right)\right) \cdot |3AryTr^n|$, where the number $|3AryTr^n|$ is given by the recursive equations of Section 4.2.1. Since $|PBFT^n| = |PBFTree^n|$ from Corollary 3, we have established the purpose of this subsection. Moreover using $(\mathrm{strip}^n)^{-1} \circ (-)^{\mathrm{rev}}|_{PBFTree^n}$ we can enumerate all the elements of $PBFT^n$.

4.3. Inductive characterization of a class of boolean functions

In this subsection, we give an inductive characterization of the class of boolean functions that consists of exactly constant functions and (positive and negative) projections.

Definition 36 (*Constant Functions*). A function f of $\overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \to \mathbb{BOOL}$ $(n \geq 1)$ is a constant function if there is $b \in \mathbb{BOOL}$ such that for any $\langle b_1, \ldots, b_n \rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n}$, $f(b_1, \ldots, b_n) = b$.

Definition 37 (*Positive and Negative Projections*). A function f of $\overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \to \mathbb{BOOL}$ $(n \geq 1)$ is a positive (respectively negative) projection if there is i $(1 \leq i \leq n)$ such that for any $\langle b_1, \ldots, b_n \rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n}$,

$$f(b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_n) = 0 \ \ (\text{resp. } f(b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_n) = 1)$$

and

$$f(b_1, \ldots, b_{i-1}, 1, b_{i+1}, \ldots, b_n) = 1 \ \ (\text{resp. } f(b_1, \ldots, b_{i-1}, 1, b_{i+1}, \ldots, b_n) = 0).$$

We define a set CP^n $(n \geq 1)$to be

$$\begin{aligned} CP^n \equiv_{\mathrm{def}} \{ f \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \to \mathbb{BOOL} \mid \\ (f \text{ is a constant function}) \vee (f \text{ is a positive or negative projection}) \}. \end{aligned}$$

Then we define $CP \equiv_{\mathrm{def}} \bigcup_{i \in \mathbb{N}} CP^i$.

We can easily see that the following proposition about permutations of CP holds.

Proposition 21. *Let $f \in CP^n$ $(n \geq 2)$. Then if we define f'_i by $f'_i(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$, then $f'_i \in CP^n$ for each i $(1 \leq i \leq n-1)$.*

On the other hand, we define CP'^{n} ($n \geq 1$) inductively as follows:

$$CP'^{1} = \{f \mid f \in \mathbb{BOOL} \to \mathbb{BOOL}\}$$

$$CP'^{n+1} = \left\{ f \mid \exists f'_1, f'_2 \in CP'^{n}.(f'_1 \text{ and } f'_2 \text{ are different constant functions}) \wedge \left((f(0, b_2, \ldots, b_{n+1}) = f'_1(b_2, \ldots, b_{n+1})) \wedge (f(1, b_2, \ldots, b_{n+1}) = f'_2(b_2, \ldots, b_{n+1})) \text{ for any } \langle b_2, \ldots, b_{n+1}\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \right) \right\}$$

$$\cup \left\{ f \mid \exists f' \in CP'^{n}.f(0, b_2, \ldots, b_{n+1}) = f'(b_2, \ldots, b_{n+2}) = f(1, b_2, \ldots, b_{n+1}) \text{ for any } \langle b_2, \ldots, b_{n+1}\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \right\}$$

We define $CP' \equiv_{\mathrm{def}} \bigcup_{i \in \mathbb{N}} CP'^{i}$.

Proposition 22 (*Inductive Characterization of CP*). $CP = CP'$.

Proof. We prove $CP^i = CP'^{i}$ ($i \in \mathbb{N}$) by induction.

(1) The case where $i = 1$:

It is obvious that $CP^1 \subseteq CP'^{1}$. Moreover CP'^{1} consists of two constant functions, one positive projection, and one negative projection. So, $CP^1 \subseteq CP'^{1}$.

(2) The case where $i > 1$:

- $CP'^{i} \subseteq CP^i$:

  We assume $f \in CP'^{i}$.

  (a) The case where there are $f'_1, f'_2 \in CP'^{i-1}$ such that f'_1 and f'_2 are different constant functions, and $f(0, b_2, \ldots, b_i) = f'_1(b_2, \ldots, b_i)$ and $f(1, b_2, \ldots, b_i) = f'_2(b_2, \ldots, b_i)$ for any $\langle b_2, \ldots, b_i\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{i-1}$: Then f is a positive or negative projection. So, $f \in CP^i$.

  (b) The case where there is $f' \in CP'^{i-1}$ such that $f(0, b_2, \ldots, b_i) = f'(b_2, \ldots, b_i) = f(1, b_2, \ldots, b_i)$ for any $\langle b_2, \ldots, b_i\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{i-1}$: Then by inductive hypothesis $f' \in CP^{i-1}$. So, f' is a constant function or (positive or negative) projection. If f' is a constant function, then f is also a constant function. If f' is a (positive or negative) projection, then f is also a projection with the same polarity as f'. So, $f \in CP^i$.

- $CP^i \subseteq CP'^{i}$:

  We assume $f \in CP^i$.

  (a) The case where f is a constant function: Then we can find a constant function $f' \in CP^{i-1}$ that returns the same value as that of f. By inductive hypothesis, $f' \in CP'^{i-1}$. Then $f(0, b_2, \ldots, b_i) = f'(b_2, \ldots, b_i) = f(1, b_2, \ldots, b_i)$ for any $\langle b_2, \ldots, b_i\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{i-1}$. So $f \in CP'^{i}$.

  (b) The case where f is a positive projection:

  First we assume that f is a positive projection on the first argument. Let $f'_1 \in CP^{i-1}$ be a constant function that always returns 0 and $f'_2 \in CP^{i-1}$ a constant function that always returns 1. By

inductive hypothesis $f_1', f_2' \in CP'^{i-1}$. Then $f(0, b_2, \ldots, b_i) = f_1'(b_2, \ldots, b_i)$ and $f(1, b_2, \ldots, b_i) = f_2'(b_2, \ldots, b_i)$ for any $\langle b_2, \ldots, b_i \rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{i-1}$. So, $f \in CP'^{i}$.
Second we assume that f is a positive projection on the j-th argument, where $2 \leq j \leq i$. On the other hand there is $f' \in CP^{i-1}$ such that f' is a positive projection on the $j-1$-th argument. By inductive hypothesis $f' \in CP'^{i-1}$. Moreover, $f(0, b_2, \ldots, b_i) = f'(b_2, \ldots, b_i) = f(1, b_2, \ldots, b_i)$ for any $\langle b_2, \ldots, b_i \rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{i-1}$. So, $f \in CP'^{i}$.

(c) The case where f is a negative projection:
Similar to the case above. □

4.4. The number of the (positive and negative) projections in $PBFT^n$

In this subsection, we count the (positive and negative) projections in $PBFT^n$. In order to do that, we need two definitions and a proposition.

Definition 38. Let $t = \lambda F_1 \ldots \lambda F_n.\lambda x.\lambda f.\lambda g.t'$ be in $PBFT^n$. Then F_i governs f and g in t if there are two paths in the tree $(t')^* \in PBFTree^n$ such that one is from F_i to f (or g respectively) through the middle outgoing edge of F_i and the other from F_i to g (resp. f) through the right outgoing edge of F_i.

Proposition 23. *If F_i governs f and g in t, then i is unique among* $\{1, \ldots, n\}$.

Proof. We assume that $F_{i'}$ governs f and g in t. Let $\leq$ be the partial order generated by the tree $(t')^*$.

(1) The case where $\neg(F_i \leq F_{i'}) \wedge \neg(F_{i'} \leq F_i)$:
There are two paths such that one is from F_i from F_i to f and the other from $F_{i'}$ to f. This contradicts $(t')^*$ being a tree.

(2) The case where $F_i \leq F_{i'}$:
Without loss of generality, we can assume that there are two paths in the tree $(t')^*$ such that one is from F_i to f through the middle outgoing edge of F_i and the other from F_i to g through the right outgoing edge of F_i. On the other hand since $F_i \leq F_{i'}$, there is a path from F_i to $F_{i'}$. Then we assume that $F_i < F_{i'}$.
   (a) The case where there is a path from F_i to $F_{i'}$ through the left outgoing edge of F_i: Then there are two paths from F_i to f in the tree $(t')^*$ such that one is through the left outgoing edge of F_i and the node $F_{i'}$ and the other through the middle outgoing edge of F_i.
   (b) The case where there is a path from F_i to $F_{i'}$ through the middle outgoing edge of F_i: Then there are two paths from F_i to g in the tree $(t')^*$ such that one is through the middle outgoing edge of F_i and the node $F_{i'}$ and the other through the right outgoing edge of F_i.
   (c) The case where there is a path from F_i to $F_{i'}$ through the right outgoing edge of F_i:
   Then there are two paths from F_i to f in the tree $(t')^*$ such that one is through the middle outgoing edge of F_i the other through the right outgoing edge of F_i and the node $F_{i'}$.

So if $F_i < F_{i'}$, then $(t')^*$ would not be a tree. Hence $i = i'$.

(3) The case where $F_{i'} \leq F_i$:
Similar to the case above. □

Definition 39. Let Θ be a closed proof net of $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ $(n \geq 1)$. Then we define setfun(Θ) to be the set-theoretic function of $\overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \to \mathbb{BOOL}$ uniquely determined by Θ only paying attention to the input–output relation on $\{\underline{0}, \underline{1}\}$ and identifying $\{\underline{0}, \underline{1}\}$ with $\mathbb{BOOL} = \{0, 1\}$.

Proposition 24. *Let $t = \lambda F_1 \ldots \lambda F_n.\lambda x.\lambda f.\lambda g.t'$ be in $PBFT^n$.*

(1) *If there is i $(1 \leq i \leq n)$ such that F_i governs f and g in t, then* setfun(t) *is a (positive or negative) projection.*
(2) *Otherwise,* setfun(t) *is a constant function.*

Proof. We prove both statements at the same time by induction on n.

- The case where $n = 1$:
  Both statements hold.
- The case where $n > 1$:
  · Proof of (1):

  Let $\leq$ be the partial order generated by the tree $(t')^*$.

  (i) The case where there is $k\,(1 \leq k \leq n)$ such that F_k does not govern f and g and is maximal among $\{F_1, \ldots, F_n\}$ in $\leq$:
  Then since **PBool** is a third-order formula and t is a linear λ-term, we can easily see that

$$\lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.((((((t\ F_1)\cdots)F_{k-1})\underline{0})F_{k+1})\cdots)F_n$$

  and

$$\lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.((((((t\ F_1)\cdots)F_{k-1})\underline{1})F_{k+1})\cdots)F_n$$

  have the same normal form. Let the normal form be

$$s = \lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.\lambda x.\lambda f.\lambda g.s'.$$

  Then since there is $j\,(1 \leq j \leq k-1 \vee k+1 \leq j \leq n)$ such that F_j governs f and g in s, by inductive hypothesis setfun(s) is a (positive or negative) projection. Then

$$\text{setfun}(t)(x_1, \ldots, x_n) = \text{setfun}(s)(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$$

  for any $\langle x_1, \ldots, x_n\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n}$. Hence setfun$(t)$ is a (positive or negative) projection.

  (ii) Otherwise:
  From Proposition 23 there is unique $k\,(1 \leq k \leq n)$ such that F_k governs f and g and is maximal among $\{F_1, \ldots, F_n\}$ in $\leq$. Then since **PBool** is a third-order formula and t is a linear λ-term, we can easily see that

$$s_1 = \lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.((((((t\ F_1)\cdots)F_{k-1})\underline{0})F_{k+1})\cdots)F_n$$

  and

$$s_2 = \lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.((((((t\ F_1)\cdots)F_{k-1})\underline{1})F_{k+1})\cdots)F_n$$

  have different normal forms. Moreover both in s_1 and s_2 for any $j\,(1 \leq j \leq k-1 \vee k+1 \leq j \leq n)$, F_j does not govern f and g. Hence by inductive hypothesis both s_1 and s_2 are different constant functions. Then setfun$(t)(x_1, \ldots, x_{k-1}, \underline{0}, x_{k+1}, x_n)$ = setfun$(s_1)(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ and setfun$(t)(x_1, \ldots, x_{k-1}, \underline{1}, x_{k+1}, x_n)$ = setfun$(s_2)(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ for any $\langle x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n-1}$. Hence setfun$(t)$ is a (positive or negative) projection.

  · Proof of (2): We assume that for any $j\,(1 \leq j \leq n)\,F_j$ does not govern f and g in t. Then we choose $k\,(1 \leq k \leq n)$ such that F_k is maximal among $\{F_1, \ldots, F_n\}$ in $\leq$ that is the partial order generated by the tree $(t')^*$. Then since **PBool** is a third-order formula and t is a linear λ-term, we can easily see that

$$\lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.((((((t\ F_1)\cdots)F_{k-1})\underline{0})F_{k+1})\cdots)F_n$$

  and

$$\lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.((((((t\ F_1)\cdots)F_{k-1})\underline{1})F_{k+1})\cdots)F_n$$

  have the same normal form. Let the normal form be

$$s = \lambda F_1.\ldots.\lambda F_{k-1}.\lambda F_{k+1}.\ldots.\lambda F_n.\lambda x.\lambda f.\lambda g.s'.$$

  Then since for any $j\,(1 \leq j \leq k-1 \vee k+1 \leq j \leq n)\,F_j$ does not govern f and g in s, by inductive hypothesis setfun(s) is a constant function. Then setfun$(t)(x_1, \ldots, x_n)$ = setfun$(s)(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ for any $\langle x_1, \ldots, x_n\rangle \in \overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n}$. Hence setfun$(t)$ is a constant function. □

Corollary 4. *$\lambda F_1 \ldots . \lambda F_n.\lambda x.\lambda f.\lambda g.t'$ be in $PBFT^n$. Then* setfun(t) *is a (positive or negative) projection iff there is i $(1 \le i \le n)$ such that F_i governs f and g in t.*

Corollary 4 gives a characterization of the set of (positive or negative) projections in $PBFT^n$. Following the characterization, we count the set. Let us call the set $PBTFProj^n$.
First we count an auxiliary set. Let $3AryTrOneU^n$ be the set of trees which are constructed from elements of $3AryTr^n$ inserting exactly one unary node. From a similar discussion to Section 4.2.2, we can easily see that $|3AryTrOneU^n| = (3n+1) \cdot |3AryTr^n|$. We note that $|3AryTrOneU^1| = (3+1) \cdot |3AryTr^1| = 4$, $|3AryTrOneU^2| = (3 \cdot 2+1) \cdot |3AryTr^2| = 7 \cdot 3 = 21$, and $|3AryTrOneU^3| = (3 \cdot 3+1) \cdot |3AryTr^3| = 10 \cdot 12 = 120$.
Next we consider the set of trees which are constructed from elements of $3AryTr^n$ inserting exactly two unary nodes labeled by f and g. Let the set be $3AryTrTwoU^n$. In an element T of $3AryTrTwoU^n$, a 3-ary node labeled by F *governs f and g* if there are two paths such that one is from F to f (or g respectively) through the middle outgoing edge of F and the other is from F to g (resp. f) through the right outgoing edge of F. We define

$$3AryTrTwoUGov^n \equiv_{\text{def}} \{T \in 3AryTrTwoU^n \mid \exists F \text{ in } T.F \text{ governs } f \text{ and } g\}.$$

Then considering permutations of $F_1, \ldots, F_n$, we can easily see that $|PBTFProj^n| = n! \cdot |3AryTrTwoUGov^n|$. Hence in the following we concentrate on obtaining $|3AryTrTwoUGov^n|$.

(1) The case where $n = 1$:
In this case any element of $3AryTrTwoUGov^1$ has height 1. It is obvious that $|3AryTrTwoUGov^1| = 2$. So $|PBTFProj^1| = 1! \cdot |3AryTrTwoUGov^1| = 2$.

(2) The case where $n = 2$:
In this case any element of $3AryTrTwoUGov^2$ has height 2. Moreover we must consider the following cases:
  - (a) The case where the root node F governs f and g:
    In this case the number of the combinations is $\binom{3}{1} \cdot |3AryTrTwoUGov^1| = 6$.
  - (b) The case where the a 3-ary node except for the root node governs f and g:
    - (i) The case where one of f and g is a son of the root node F: In this case the number of the combinations is $2 \cdot 2 \cdot |3AryTrOneU^1| = 16$.
    - (ii) The case where both f and g are sons of the root node F: In this case the number of the combinations is $2 \cdot |3AryTr^1| = 2$.

  Summing up, we obtain $|3AryTrTwoUGov^2| = 6+16+2 = 24$. Hence, $|PBTFProj^2| = 2! \cdot |3AryTrTwoUGov^2| = 48$.

(3) The case where $n = 3$:
  - (a) The case where exactly two sons of the root node F are leaves:
    In this case we can do a similar case-analysis to the case where $n = 2$. The number of the combinations is $\binom{3}{1} \cdot |3AryTrTwoUGov^2| + 2 \cdot 2 \cdot |3AryTrOneU^2| + 2 \cdot |3AryTr^2| = 3 \cdot 8 + 4 \cdot 21 + 2 \cdot 3 = 114$.
  - (b) The case where exactly one son of the root node F is a leaf:
    - (i) The case where a 3-ary node except for the root node governs f and g:
      The number of the combinations is
      $3! \cdot |3AryTrTwoUGov^1| \cdot |3AryTr^1| = 12$.
    - (ii) The case where the root node F governs f and g:
      - (A) The case where neither f nor g is a son of the root node F:
        The number of the combinations is $2 \cdot |3AryTrOneU^1| \cdot |3AryTrOneU^1| = 32$.
      - (B) The case where one of f and g is a son of the root node F: The number of the combinations is $2 \cdot 2 \cdot |3AryTrOneU^1| \cdot |3AryTr^1| = 16$.

  Summing up, we obtain $|3AryTrTwoUGov^3| = 114 + 12 + 32 + 16 = 174$. So $|PBTFProj^3| = 3! \cdot |3AryTrTwoUGov^3| = 1044$.

(4) The case where $n \ge 4$:

(a) The case where exactly two sons of the root node F are leaves:
In this case we can do a similar case-analysis to the case where $n = 2$. The number of the combinations is

$$\binom{3}{1} \cdot |3AryTrTwoUGov^{n-1}| \\ + 2 \cdot 2 \cdot |3AryTrOneU^{n-1}| \\ + 2 \cdot |3AryTr^{n-1}|.$$

(b) The case where exactly one son of the root node F is a leaf:
In this case we can do a similar case-analysis to subcase (3b) of the case where $n = 3$. The number of the combinations is

$$3! \cdot \sum_{k+\ell=n-1,k,\ell\geq 1} (|3AryTrTwoUGov^{k}| \cdot |3AryTr^{\ell}|) \\ + 2 \cdot \sum_{k+\ell=n-1,k,\ell\geq 1} (|3AryTrOneU^{k}| \cdot |3AryTrOneU^{\ell}|) \\ + 2 \cdot 2 \cdot \sum_{k+\ell=n-1,k,\ell\geq 1} (|3AryTrOneU^{k}| \cdot |3AryTr^{\ell}|).$$

(c) The case where any of the root node F is not a leaf: The number of the combinations is

$$2 \cdot \sum_{k+\ell+m=n-1,k,\ell,m\geq 1} (|3AryTr^{k}| \cdot |3AryTrOneU^{\ell}| \cdot |3AryTrOneU^{m}|).$$

4.5. The class of functions represented by the elements in PBFT

Let Θ be a closed proof net of $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ $(n \geq 1)$. We define $PBFF^n \equiv_{\text{def}} \{\text{setfun}(\Theta) \mid \Theta \in PBFT^n\}$ for n $(n \geq 1)$ and $PBFF \equiv_{\text{def}} \{\text{setfun}(\Theta) \mid \Theta \in PBFT\}$.

Theorem 3. $PBFF = CP'$.

Proof. We prove that $PBFF^n = CP'^n$ by induction on n.

(1) The case where $n = 1$:
Both $PBFF^1$ and CP'^1 are the set of all the functions of $\{0, 1\} \to \{0, 1\}$. So, $PBFT^1 = CP'^1$.

(2) The case where $n > 1$:

(a) $CP'^n \subseteq PBFF^n$:
Let $f \in CP'^n$.
(i) The case where there exist $f_1', f_2' \in CP'^{n-1}$ such that f_1' and f_2' are different constant functions and $f(0, b_2, \ldots, b_n) = f_1'(b_2, \ldots, b_n)$ and $f(1, b_2, \ldots, b_n) = f_2'(b_2, \ldots, b_n)$:
Then let t be $\lambda F_1.\ldots\lambda F_n.\lambda x.\lambda f.\lambda g.F_n(F_{n-1}(\cdots(F_1 x(\lambda x.fx)(\lambda x.gx))\cdots)II)II$ or $\lambda F_1.\ldots\lambda F_n.\lambda x.\lambda f.\lambda g.F_n(F_{n-1}(\cdots(F_1 x(\lambda x.gx)(\lambda x.fx))\cdots)II)II$ depending on which constant functions f_1' and f_2' are, where I is $\lambda x.x$. Then it is obvious that $\text{setfun}(t) = f$.
(ii) The case where there exists $f' \in CP'^{n-1}$ such that $f(0, b_2, \ldots, b_{n+1}) = f'(b_2, \ldots, b_{n+2}) = f(1, b_2, \ldots, b_{n+1})$:
By inductive hypothesis, there exists $t' \in FBFT^{n-1}$ such that $\text{setfun}(t') = f'$. Then t' must have the form $\lambda F_1.\ldots\lambda F_{n-1}.\lambda x.\lambda f.\lambda g.t''$. Let t be $\lambda F_n.\lambda F_1.\ldots\lambda F_{n-1}.\lambda x.\lambda f.\lambda g.F_n x I(\lambda x.t'')$. It is obvious that $\text{setfun}(t) = f$.

(b) $PBFF^n \subseteq CP'^n$:
From Propositions 22 and 24, $PBFF^n \subseteq CP^n = CP'^n$. $\square$

Corollary 5. *The functions represented by the closed proof nets of* $\overbrace{\mathbf{PBool} \multimap \cdots \mathbf{PBool}}^{n} \multimap \mathbf{PBool}$ $(n \geq 1)$ *are exactly constant functions and (positive and negative) projections of* $\overbrace{\mathbb{BOOL} \times \cdots \times \mathbb{BOOL}}^{n} \to \mathbb{BOOL}$.

5. Separation for IIMLL proof nets with order less than 4

In this section we give a separation result about IIMLL closed proof nets with order less than 4. Moreover we state the main theorem in this paper.

We cannot perform the separation directly, i.e., simply applying a wrapping net to two different IMLL closed proof nets with the same positive conclusion. We need type instantiation.

Definition 40 (*Type Instantiation*). Let Θ be an IIMLL proof net and A be an MLL formula. The type instantiated proof net $\Theta[A/p]$ of Θ w.r.t. A is an IIMLL proof net obtained from Θ by replacing each atomic formula occurrence p by A.

We note that $\Theta[A/p]$ is not generally a normal form and in order to obtain the normal form multiplicative η-expansion must be applied to $\Theta[A/p]$ several times (but, we identify $\Theta[A/p]$ with the normal form as we said before). Moreover we note that in discussions of the last section, we have already stated that our choice of A is $\mathbf{PBool} = p \multimap (p \multimap p) \multimap (p \multimap p) \multimap p$.

First we reduce the separation problem of IMLL closed proof nets with order less than 4 to that of the second-order linear term system.

Definition 41 (*The Second-order Linear Term System*). (1) The language:

(a) A denumerable set of variables *Var*:

Elements of *Var* are denoted by $x_1, x_2, \ldots$.

(b) A denumerable set of second-order variables *SVar*:

Elements of *SVar* are denoted by $G_1, G_2, \ldots$. Moreover each element G of *SVar* has its arity $\text{arity}(G) \geq 1$.

(2) The set SLT of the terms of the language and $FV : SLT \to \mathbb{P}(Var \cup SVar)$ is inductively as follows (where $\mathbb{P}(Var \cup SVar)$ is the set of all subsets of $Var \cup SVar$):

(a) If $x \in Var$, then $x \in SLT$ and $FV(x) = \{x\}$;

(b) If $\{t_1, \ldots, t_n\} \subseteq SLT$, $G \in SVar$ has the arity $\text{arity}(G) = n$, for each i, j ($1 \leq i, j \leq n$), when $i \neq j$, $FV(t_i) \cap FV(t_j) = \emptyset$, and for each i ($1 \leq i \leq n$), $G \notin FV(t_i)$, then $G(t_1, \ldots, t_n) \in SLT$ and $FV(G(t_1, \ldots, t_n)) = \{G\} \cup \bigcup_{1 \leq i \leq n} FV(t_i)$.

(3) Assignments:

(a) A variable assignment is a function $\rho_1 : Var \to \mathbb{BOOL}$.

(b) A second-order variable assignment is a function $\rho_2 : SVar \to CP$ such that if $G \in SVar$ and $\text{arity}(G) = n$, then $\rho_2(G) \in CP^n$.

(4) Models:

A model for SLT $[|-|]_{\langle \rho_1, \rho_2 \rangle} : SLT \to \mathbb{BOOL}$ is determined uniquely for a given $\langle \rho_1, \rho_2 \rangle$ as follows:

(a) $[|x|]_{\langle \rho_1, \rho_2 \rangle} = \rho_1(x)$.

(b) $[|G(t_1, \ldots, t_n)|]_{\langle \rho_1, \rho_2 \rangle} = \rho_2(G)([|t_1|]_{\langle \rho_1, \rho_2 \rangle}, \ldots, [|t_n|]_{\langle \rho_1, \rho_2 \rangle})$.

We note that in the definition above to each second-order variable an element of *CP* is assigned, i.e., a constant function or a (positive or negative) projection.

Proposition 25. *Let s, t be in SLT. If $s \neq t$, then there are a variable assignment ρ_1 and a second-order variable assignment ρ_2 such that $[|s|]_{\langle \rho_1, \rho_2 \rangle} \neq [|t|]_{\langle \rho_1, \rho_2 \rangle}$.*

Proof. First, to each variable x of *Var* we assign a leaf labeled by x of Fig. 39. Similarly, to each second-order variable G with arity n of *SVar* we assign an n-ary node labeled by G of Fig. 39. Then the set *SLTTree* of the finite trees consisting of only such nodes is defined in a usual manner. Then it is obvious that there is a bijection $(-)^* : SLT \to SLTTree$.

Next we prove this proposition by induction on $\text{hght}((s)^*) + \text{hght}((t)^*)$.

(1) The case where $\text{hght}((s)^*) + \text{hght}((t)^*) = 0$:

Then $s = x_i \neq x_j = t$. We just choose ρ_1 such that $\rho_1(x_i) = 0$ and $\rho_1(x_j) = 1$. Hence $[|s|]_{\langle \rho_1, \rho_2 \rangle} = \rho_1(x_i) = 0 \neq 1 = \rho_1(x_j) = [|t|]_{\langle \rho_1, \rho_2 \rangle}$.

(2) The case where $\text{hght}((s)^*) + \text{hght}((t)^*) > 0$:

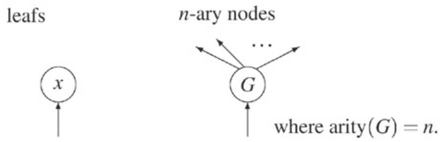


Fig. 39. Nodes for *SLTTree*.

(a) The case where $s = x$:
Then t must have the form $G(t_1, \ldots, t_n)$. We choose ρ_1 such that $\rho_1(x) = 0$. Moreover we choose ρ_2 such that $\rho_2(G) \in CP^n$ is the constant function that always returns 1. Then

$$\begin{aligned}[|s|]_{\langle\rho_1,\rho_2\rangle} &= \rho_1(x)\\ &= 0 \neq 1\\ &= \rho_2(G)([|t_1|]_{\langle\rho_1,\rho_2\rangle}, \ldots, [|t_n|]_{\langle\rho_1,\rho_2\rangle})\\ &= [|G(t_1, \ldots, t_n)|]_{\langle\rho_1,\rho_2\rangle}\\ &= [|t|]_{\langle\rho_1,\rho_2\rangle}.\end{aligned}$$

(b) The case where $t = x$:
Similar to the case above.

(c) Otherwise:
Then $s = G_k(s_1, \ldots, s_m)$ and $t = G_\ell(t_1, \ldots, t_n)$.

(i) The case where $G_k \neq G_\ell$: Then we choose ρ_2 such that $\rho_2(G_k)$ and $\rho_2(G_\ell)$ are the constant functions that always return 0 and 1 respectively. Then

$$\begin{aligned}[|s|]_{\langle\rho_1,\rho_2\rangle} &= [|G_k(s_1, \ldots, s_m)|]_{\langle\rho_1,\rho_2\rangle}\\ &= \rho_2(G_k)([|s_1|]_{\langle\rho_1,\rho_2\rangle}, \ldots, [|s_m|]_{\langle\rho_1,\rho_2\rangle})\\ &= 0 \neq 1\\ &= \rho_2(G_\ell)([|t_1|]_{\langle\rho_1,\rho_2\rangle}, \ldots, [|t_n|]_{\langle\rho_1,\rho_2\rangle})\\ &= [|G_\ell(t_1, \ldots, t_n)|]_{\langle\rho_1,\rho_2\rangle}\\ &= [|t|]_{\langle\rho_1,\rho_2\rangle}.\end{aligned}$$

(ii) Otherwise: Then $s = G(s_1, \ldots, s_n)$ and $t = G(t_1, \ldots, t_n)$. Since $s \neq t$, there is a i such that $s_i \neq t_i$. By inductive hypothesis, there are ρ_1 and ρ_2 such that $[|s_i|]_{\langle\rho_1,\rho_2\rangle} \neq [|t_i|]_{\langle\rho_1,\rho_2\rangle}$. Let prj_i^n be the i-th positive projection in CP^n. Then

$$\begin{aligned}&[|s|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}\\ &\quad= [|G(s_1, \ldots, s_n)|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}\\ &\quad= \rho_2[G \mapsto prj_i^n](G)([|s_1|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}, \ldots, [|s_n|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle})\\ &\quad= prj_i^n([|s_1|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}, \ldots, [|s_n|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle})\\ &\quad= [|s_i|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}\\ &\quad\neq [|t_i|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}\\ &\quad= prj_i^n([|t_1|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}, \ldots, [|t_n|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle})\\ &\quad= \rho_2[G \mapsto prj_i^n](G)([|t_1|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n\rangle}, \ldots, [|t_n|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle})\\ &\quad= [|G(t_1, \ldots, t_n)|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}\\ &\quad= [|t|]_{\langle\rho_1,\rho_2[G\mapsto prj_i^n]\rangle}. \quad \square\end{aligned}$$

Let *CNPNLess*4 be the set of all IIMLL closed normal proof nets with order less than 4. Next, we define an injection $(-)^\bullet$ from *CNPNLess*4 to SLT. We note that a positive IIMLL formula A^+ with order less than 4 having at least one closed proof net must have the form $B_1 \multimap \cdots \multimap B_n \multimap C_1 \multimap \cdots \multimap C_m \multimap p^+$, where

(1) $n \geq 0$ and $m \geq 1$ (which is justified by Lemma 13).

(2) $B_i = \overbrace{p \multimap \cdots \multimap p}^{k_i} \multimap p$ $(k_i \geq 1)$ for each i $(1 \leq i \leq n)$.
(3) $C_j = p$ for each j $(1 \leq j \leq m)$.

(Recall that we identify A^+ with the equivalence class induced by $\equiv$ (Definition 31). So, we can normalize A^+ to the form given above.)

Let Θ in *CNPNLess*4 be with the positive conclusion A^+. Then term(Θ) has the form $\lambda G_1.\ldots\lambda G_n.\lambda x_1.\ldots.\lambda x_m.t'$, where t' is not a λ-abstraction and $FV(t') = \{G_1, \ldots, G_n, x_1, \ldots, x_n\}$. Moreover G_i has the type $B_i = \overbrace{p \multimap \cdots \multimap p}^{k_i} \multimap p$ for each i $(1 \leq i \leq n)$ and x_j has the type $C_j = p$ for each j $(1 \leq j \leq m)$. By identifying each variable G_i (resp. x_j) in Θ with second-order variable G_i with arity k_i (resp. variable x_j) of the second-order linear term system respectively, t' can be regarded as a term of SLT. Then we define $(\Theta)^\bullet = t'$.

Proposition 26. *The function* $(-)^\bullet : CNPNLess4 \to SLT$ *is injective.*

Proof. Let Θ_1, Θ be in *CNPNLess*4 such that $\Theta_1 \neq \Theta_2$.

(1) The case where the positive conclusion of Θ_1 is different from that of Θ_2:
Then since $FV((\Theta_1)^\bullet) \neq FV((\Theta_2)^\bullet)$, $(\Theta_1)^\bullet \neq (\Theta_2)^\bullet$.
(2) The case where the positive conclusion of Θ_1 is the same as that of Θ_2:
Then from Proposition 17 term$(\Theta_1) \neq$ term(Θ_2). Then term(Θ_1) and term(Θ_2) have the forms $\lambda G_1.\ldots\lambda G_n.\lambda x_1.\ldots.\lambda x_m.t'_1$ and $\lambda G_1.\ldots\lambda G_n.\lambda x_1.\ldots.\lambda x_m.t'_2$ respectively such that $t'_1 \neq t'_2$. Hence $(\Theta_1)^\bullet = t'_1 \neq t'_2 = (\Theta_2)^\bullet$. $\square$

Proposition 27. *Let* Θ *with the positive conclusion*

$A^+ = B_1 \multimap \cdots \multimap B_n \multimap C_1 \multimap \cdots \multimap C_m \multimap p^+$ *be in CNPNLess*4*. If* $[|(\Theta)^\bullet|]_{\langle \rho_1, \rho_2 \rangle} = b$ *for an assignment pair* $\langle \rho_1, \rho_2 \rangle$*, then there is a wrapping net* $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ *such that* $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta[\mathbf{PBool}/p]] = \underline{b}$*.*

Proof. Let term(Θ) be $\lambda G_1.\ldots\lambda G_n.\lambda x_1.\ldots.\lambda x_m.t'$, where t' is not a λ-abstraction. Then $(\Theta)^\bullet = t'$ and $FV(t') = \{G_1 \ldots, G_n, x_1, \ldots, x_m\}$.

We can prove that there is a wrapping net $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ with the form of Fig. 40 such that setfun$(C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta[\mathbf{PBool}/p]]) = [|(\Theta)^\bullet|]_{\langle \rho_1, \rho_2 \rangle}$ by induction on the height $((\Theta)^\bullet)^*$:

(1) The case where hght$(((\Theta)^\bullet)^*) = 0$:
Then term$(\Theta) = \lambda x_1.x_1$ and $A^+ = p \multimap p^+$. From the assumption we find a closed IIMLL proof net Π with the conclusion $\mathbf{PBool}^+$ such that $\underline{[|(\Theta)^\bullet|]_{\langle \rho_1, \rho_2 \rangle}} = \underline{\rho_1(x_1)} = \Pi$. Then let $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ be the IIMLL proof net shown in Fig. 41. It is obvious that setfun$(C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta[\mathbf{PBool}/p]]) =$ setfun$(\Pi) = [|(\Theta)^\bullet|]_{\langle \rho_1, \rho_2 \rangle}$ and $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ has the form of Fig. 40.
(2) The case where hght$(((\Theta)^\bullet)^*) > 1$:
Then $(\Theta)^\bullet$ has the form $G_{i'}(t'_1, \ldots, t'_h)$ for some i' $(1 \leq i' \leq n)$. Since $\{t'_1, \ldots, t'_h\} \subseteq SLT$, we can find $\{\Lambda_1, \ldots, \Lambda_h\} \subseteq CNPNLess4$ such that $(\Lambda_\ell)^\bullet = t'_\ell$ for each j' $(1 \leq \ell \leq h)$. By inductive hypothesis, there are wrapping nets $C_\ell[]$ with the form of Fig. 40 such that setfun$(C_\ell[\Lambda_\ell[\mathbf{PBool}/p]]) = [|(\Lambda_\ell)^\bullet|]_{\langle \rho_1, \rho_2 \rangle}$ $(1 \leq \ell \leq h)$. Since $(\Lambda_\ell)^\bullet = t'_\ell$ and t'_ℓ are linear terms $(1 \leq \ell \leq h)$, we find a closed proof net Θ_i with the positive conclusion $B_i[\mathbf{PBool}/p]^+$ that is a subproof net of $C_\ell[]$ for some ℓ $(1 \leq \ell \leq h)$ such that setfun$(\Theta_i) = \rho_2(G_i)$ for each i $(1 \leq i \leq n, i \neq i')$. Similarly we find a closed proof net Π_j with the positive conclusion $\mathbf{PBool}^+$ that is a subproof net of $C_\ell[]$ for some ℓ $(1 \leq \ell \leq h)$ such that $\underline{\rho_1(x_j)} = \Pi_j$ for each j $(1 \leq j \leq m)$. Moreover from Corollary 5 we can find a closed IIMLL proof net $\Theta_{i'}$ with the positive conclusion $B_{i'}[\mathbf{Pbool}/p]^+$ such that setfun$(\Theta_{i'}) = \rho_2(G_{i'})$. Using these $\{\Theta_1, \ldots, \Theta_n\}$ and $\{\Pi_1, \ldots, \Pi_m\}$, we can construct a wrapping net $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ shown in Fig. 40. Then,

$$\begin{aligned}
&\text{setfun}(C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta[\mathbf{PBool}/p]]) \\
&\quad = \text{setfun}(\Theta_{i'})(\text{setfun}(C_1[\Lambda_1[\mathbf{PBool}/p]]), \ldots, \text{setfun}(C_h[\Lambda_h[\mathbf{PBool}/p]])
\end{aligned}$$

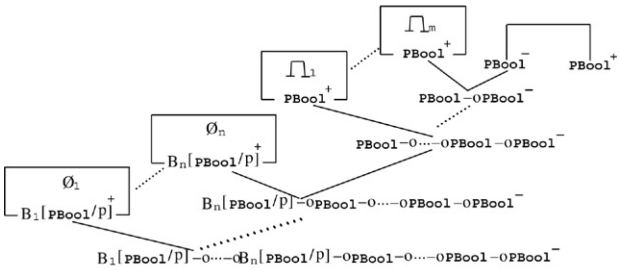


Fig. 40. A wrapping net.

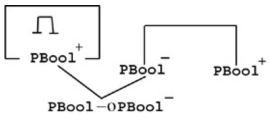


Fig. 41. A wrapping net.

$$
\begin{aligned}
&= \mathrm{setfun}(\Theta_{i'})([|(\Lambda_1)^{\bullet}|]_{\langle\rho_1,\rho_2\rangle}, \ldots, [|(\Lambda_h)^{\bullet}|]_{\langle\rho_1,\rho_2\rangle}) \text{ (IH)} \\
&= \mathrm{setfun}(\Theta_{i'})([|t'_1|]_{\langle\rho_1,\rho_2\rangle}, \ldots, [|t'_h|]_{\langle\rho_1,\rho_2\rangle}) \\
&= \rho_2(G_{i'})([|t'_1|]_{\langle\rho_1,\rho_2\rangle}, \ldots, [|t'_h|]_{\langle\rho_1,\rho_2\rangle}) \\
&= [|G_{i'}(t'_1, \ldots, t'_h)|]_{\langle\rho_1,\rho_2\rangle} \\
&= [|\Theta^{\bullet}|]_{\langle\rho_1,\rho_2\rangle}. \quad \square
\end{aligned}
$$

Note that the wrapping net $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ of the proposition above only depends on A^+ and $\langle\rho_1, \rho_2\rangle$, not on Θ.

Theorem 4. *Let Θ_1, Θ_2 with the same positive conclusion A^+ be in CNPNLess4. If $\Theta_1 \neq \Theta_2$, then there is a wrapping net $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}$ such that $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_1[\mathbf{PBool}/p]] \neq C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_2[\mathbf{PBool}/p]]$.*

Proof. Since $\Theta_1 \neq \Theta_2$, from Proposition 26 $(\Theta_1)^{\bullet} \neq (\Theta_2)^{\bullet}$. Then from Proposition 25 there are a variable assignment ρ_1 and a second-order variable assignment ρ_2 such that $[|(\Theta_1)^{\bullet}|]_{\langle\rho_1,\rho_2\rangle} \neq [|(\Theta_2)^{\bullet}|]_{\langle\rho_1,\rho_2\rangle}$. Then from Proposition 27 there is a wrapping net $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ such that

$$C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_1[\mathbf{PBool}/p]] \neq C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_2[\mathbf{PBool}/p]]. \quad \square$$

Corollary 6 (*Weak Typed Böhm Theorem on IIMLL*). *Let Θ_1 and Θ_2 be closed IIMLL proof nets with the same positive conclusion A^+ such that $\Theta_1 \neq \Theta_2$. Then there is a context $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$ such that*

$$C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_1[\mathbf{PBool}/p]] \neq C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_2[\mathbf{PBool}/p]].$$

Proof. By Theorem 2, Corollary 6, and Lemma 3. $\square$

Example 6. We explain the results of this section in terms of two IMLL proof nets Θ_1^d and Θ_1^d of Figs. 32 and 33. We recall $\Theta_1^d \neq \Theta_1^d$. Then $(\Theta_1^d)^{\bullet} = G_1(x_4, x_2, x_1, x_3)$ and $(\Theta_2^d)^{\bullet} = G_1(x_4, x_1, x_2, x_3)$. It is obvious that $(\Theta_1^d)^{\bullet} \neq (\Theta_2^d)^{\bullet}$, since although the root G_1 of $((\Theta_1^d)^{\bullet})^*$ is the same as that of $((\Theta_1^d)^{\bullet})^*$, the second argument x_2 of G_1 in$(\Theta_1^d)^{\bullet}$ is not the same as that of G_1 in$(\Theta_1^d)^{\bullet}$, i.e., x_1. Following Proposition 25 we choose a variable assignment ρ_1 such that $\rho_1(x_1) = 0$ and $\rho_1(x_2) = 1$. Moreover we choose a second-order variable assignment ρ_2 such that $\rho_2(G_1)$ is the positive projection w.r.t. the second argument. Then $[|(\Theta_1^d)^{\bullet}|]_{\langle\rho_1,\rho_2\rangle} = 0 \neq 1 = [|(\Theta_2^d)^{\bullet}|]_{\langle\rho_1,\rho_2\rangle}$. Following Proposition 27, we can find a wrapping context $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[]$, where A^+ is the positive conclusion of Θ_1^d and Θ_2^d, such that $C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_1^d] \neq C^{A[\mathbf{PBool}/p]^+}_{\mathbf{PBool}^+}[\Theta_2^d]$ (we omit this).

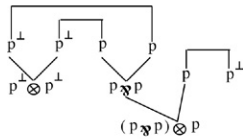


Fig. 42. A counterexample.

6. Concluding remarks

Our result is easily extendable to IMLL *with the multiplicative unit* **1** under a reasonable equality on the extended system, because the multiplicative unit can be considered as a degenerated IMLL formula. For example $\mathbf{1}^+$ has just one closed proof net and the closed proof nets on $\mathbf{1} \multimap p \multimap (p \multimap p) \multimap (p \multimap p) \multimap p^+$ have almost the same behaviour as that of $p \multimap (p \multimap p) \multimap (p \multimap p) \multimap p^+$. However, our separation result w.r.t. IMLL with **1** is stated as follows:

> Let Θ_1 and Θ_2 be closed IMLL with **1** proof nets with the same positive conclusion such that $\Theta_1 \neq \Theta_2$. Then there is a context $C[]$ such that $C[\Theta_1]$ and $C[\Theta_2]$ are closed proof nets of $\mathbf{1} \multimap \mathbf{1}^+$ and $C[\Theta_1] \neq C[\Theta_2]$.

There are two closed normal proof nets of $\mathbf{1} \multimap \mathbf{1}^+$: one consists of exactly three links (an axiom link for $\mathbf{1}^+$, a weakening link for $\mathbf{1}^-$, and a $\invamp$-link). Let the proof net be $\underline{\mathtt{ff}}_{\mathbf{1} \multimap \mathbf{1}^+}$. The other consists of exactly two links (an ID-link with $\mathbf{1}^-$ and $\mathbf{1}^+$ and a $\invamp$-link). Let the proof net be $\underline{\mathtt{tt}}_{\mathbf{1} \multimap \mathbf{1}^+}$. The proof is similar to that of IMLL without **1**.

However in a symmetric monoidal closed category (SMCC, for example, see [12]), $\underline{\mathtt{ff}}_{\mathbf{1} \multimap \mathbf{1}^+}$ and $\underline{\mathtt{tt}}_{\mathbf{1} \multimap \mathbf{1}^+}$ are interpreted into the same arrow id_I, where I is the multiplicative unit of a SMCC. To avoid such an identification, it is possible to relax the conditions of SMCC: one is to remove the axiom $l_I = r_I$. The other is that we do not assume I is isomorphic to $I \otimes I$; we just assume that I is a retract of $I \otimes I$, that is, we remove two axioms $l_A ; l_A{}^{-1} = id_{I \otimes A}$ and $r_A ; r_A{}^{-1} = id_{A \otimes I}$. The relaxation is quite natural: for example, without these axioms we can derive important equations like $\alpha_{I,A,B} ; l_{A \otimes B} = l_A \otimes id_B$. In the relaxed SMCC, proof nets of IMLL with **1** can be an internal language.

On the other hand, our result cannot be extended to classical multiplicative Linear Logic (for short MLL) directly, because all MLL proof nets cannot be polarized by the IMLL polarity. For example, the MLL proof net of Fig. 42 cannot be transformed to an IMLL proof net by type instantiation.

As an another direction, fragments including additive connectives may be studied. Currently it is proved that our method can be applied to a restricted fragment of intuitionistic multiplicative additive linear logic. The restriction is as follows:

(1) With-formulas must positively occur only as $A \& A$;
(2) Plus-formulas must negatively occur only as $A \oplus A$.

Moreover we can also prove the strong statement of the typed Böhm theorem w.r.t. the fragment. Our ongoing work is to eliminate the restriction.

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